Capacity Planning of a Perinatal Network with Generalised Loss Network Model with Overflow

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Abstract

Recent literature shows that the arrival and discharge processes in hospital intensive care units do not satisfy the Markovian property, that is, interarrival times and length of stay tend to have a long tail. In this paper we develop a generalised loss network framework for capacity planning of a perinatal network in the UK. Decomposing the network by hospitals, each unit is analysed with a GI/G/c/0 overflow loss network model. A two-moment approximation is performed to obtain the steady state solution of the GI/G/c/0 loss systems, and expressions for rejection probability and overflow probability have been derived. Using the model framework, the number of required cots can be estimated based on the rejection probability at each level of care of the neonatal units in a network. The generalisation ensures that the model can be applied to any perinatal network for renewal arrival and discharge

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processes.

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1. Introduction

In most of the developed world neonatal care has been organised into networks of cooperating hospitals (units) in order to provide better and more efficient care for the local population. A neonatal or perinatal network in the UK offers all ranges of neonatal care referred to as intensive, high dependency and special care through level 1 to level 3 units. Recent studies show that perinatal networks in the UK have been struggling with severe capacity crisis (Bliss, 2007; National Audit Office, 2007). Expanding capacity by number of beds in the unit, in general, is not an option since neonatal care is an unusually expensive therapy. Reducing capacity is not an option either, as this would risk sick neonates being denied admission to the unit or released prematurely. Consequently, determining cot capacity has become a major concern for perinatal network managers in the UK. We have addressed this issue in several papers (Asaduzzaman and Chaussalet, 2008; Asaduzzaman et al., 2010a,b; Asaduzzaman, 2010; Asaduzzaman and Chaussalet, 2011). However, as noted by Asaduzzaman et al. (2010a) and Asaduzzaman (2010) the mean is much lower than standard deviation for interarrival times and length of stay (LOS). Clearly this is a violation of the property of the exponential distribution, and hence of the Markovian assumption. Asaduzzaman (2010) has reported that the graphs and diagrams of the interarrival and LOS show

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a non-exponential pattern, and that a mixture of distributions would provide a better fit. Moreover, several mathematical (Vasilakis and Marshall, 2005; Griffiths et al., 2006; Asaduzzaman, 2010) and medical (Weissman, 1997, 2000; Rapoport et al., 2003) studies also show that the length of stay is heavily skewed, i.e. has a long tail, in ICU in particular.

Queueing models having zero buffer also referred to as 'loss models' (./././0) have been widely applied in hospital systems and intensive care in particular (e.g., Van Dijk and Kortbeek, 2009; Litvak et al., 2008; Asaduzzaman et al., 2010a,b; Asaduzzaman and Chaussalet, 2011). Van Dijk and Kortbeek (2009) proposed an M/M/c/0 loss model for capacity management in an Operating Theatre-Intensive Care Unit. Litvak et al. (2008) developed an overflow model with loss framework for capacity planning in intensive care units while Asaduzzaman et al. (2010a,b) developed a loss network model for a neonatal unit, and extended the model framework to a perinatal network in Asaduzzaman and Chaussalet (2011). These models assume that inter-arrival times and length of stay follow exponential distributions.

Queueing models with exponential inter-arrival and service times are easiest to study, since such processes are Markov chains. However, length of stay distribution in intensive care may be highly skewed (Griffiths et al., 2006). Performance measures of a queueing system with non-zero buffer are insensitive to service time distribution provided that the arrival process is Poisson (Kelly, 1979). This insensitivity property is, in general, no longer valid in the case of zero buffer or loss systems (Klimenok et al., 2005).

Many approaches have been found towards generalising such processes since Erlang introduced the M/M/c/0 model for a simple telephone network

and derived the well-known loss formula that carries his name in 1917 (Kelly, 1991; Whitt, 2004). Takács (1956, 1962) considered the loss system with general arrival pattern (GI/M/c/0) through Laplace transform. Recently there has been a growing interest in loss systems where both arrival and service patterns are generalised (GI/G/c/0). The theoretical investigation of the GI/G/c/0 loss model through the theory of random point processes has attracted many researchers. Brandt and Lisek (1980) gave a method for approximating the GI/GI/c/0 queue by means of the GI/GI/∞ queue, while Whitt (1984) applied a similar approximation under heavy traffic. Franken et al. (1982) examined the continuity property of the model, and established an equivalence between arrival and departure probability. Miyazawa and Tijms (1993) gave an approximation method for the batch-arrival GI$^{[x]}$/G/c/N queue which is applicable when the traffic intensity is less than one. The M/G/c/N and the GI/G/c/N queue have also been studied widely; for a comparison of methods, see Kimura (2000). Although many studies have been found in the literature, no simple expression for the steady state distribution is available for a GI/G/c/0 system. Hsin and Van de Liefvoort (1996) provided the exact solution for the GI/GI/c/0 system expressing the inter-arrival and service time by matrix exponential distribution. The method is computationally intensive and often includes imaginary components in the expression (which are unrealistic). Diffusion approximations, which require complicated Laplace transforms have also been used for analysing GI/G/c/N queues (e.g., Kimura, 2003; Whitt, 2004). Kim and Chae (2003) derived a transform-free expression for the analysis of the GI/G/1/N queue through the decomposed Little's formula. A two-moment approximation was proposed to

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estimate the steady state queue length distribution. Using the same approximation, Choi et al. (2005) extended the system for the multi-server finite buffer queue based on the system equations derived by Franken et al. (1982). Atkinson (2009) developed a heuristic approach for the numerical analysis of GI/G/c/0 queueing systems with examples of the two-phase Coxian distribution. Recently, Izady and Worthington (2011) performed an approximate analysis of time-dependent loss queues and networks of loss queues where general service time distributions are considered.

In this paper we derive a generalised loss network model with overflow for a network of neonatal hospitals extending the results obtained by Franken et al. (1982). Since some model parameters cannot be computed practically, a two-moment based approximation method is applied for the steady state analysis as proposed by Kim and Chae (2003). The model is then applied to the north central London Perinatal network, one of the busiest network in the UK. Data obtained from each hospital (neonatal unit) of the network have been used to check the performance of the model. The rest of the paper is organised as follows: in the next section we first discuss a typical perinatal network and then develop a generalised loss model with overflow for the network. The steady state distribution and expression for rejection and overflow probabilities have been derived for each level of care of the neonatal units. Application of the model and numerical results are presented in Section 4.

2. Structure of a perinatal network

A perinatal network in the UK is organised through level 1, level 2 and level 3 units. Figure 1 shows a typical perinatal network in the UK. Level 1 units consist of a special care baby unit (SCBU). It provides only special care which is the least intensive and most common type of care. In these units, neonates may be fed through a tube, supplied with extra oxygen or treated with ultraviolet light for jaundice. Figure 2 shows the typical patient flow in a level 1 unit. A level 1 unit may also have an intensive therapy unit (ITU) which provides short-term intensive care to neonates, and the unit may then be referred to as 'level 1 unit with ITU'. Figure 3 shows the structure of a level 1 unit with ITU. Level 2 units consist of a SCBU and a HDU where neonates can receive high dependency care such as breathing via continuous positive airway pressure or intravenous feeding. These units may also provide short-term intensive care. A level 3 unit provides all ranges of neonatal care and consists of an SCBU, an HDU and an NICU where neonates will often be on a ventilator and need constant care to be kept alive. Level 2 and level 3 units may also have some transitional care (TC) cots, which may be used to tackle overflow and rejection from SCBU. Although level 2 and level 3 units have similar structures level 2 units might not have sufficient clinician support for the NICU. NICU are HDU are often merged in level 2 and level 3 units for higher utilisation of cots. In level 2 or level 3 units, NICU-HDU neonates are sometimes initially cared at SCBU when all NICU cots are occupied. Similarly SCBU neonates are cared at NICU-HDU or TC, depending upon the availability of cots, staff and circumstances. This temporary care is provided by staffing a cot with appropriate nurse and
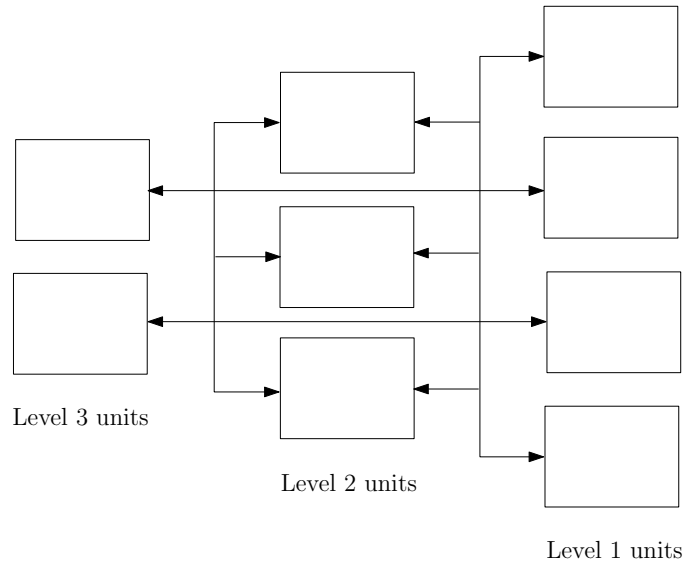
Level 3 units

Level 2 units

Level 1 units

Figure 1: Topology of a typical perinatal network. The arrows indicate forward and backward transfers between units.
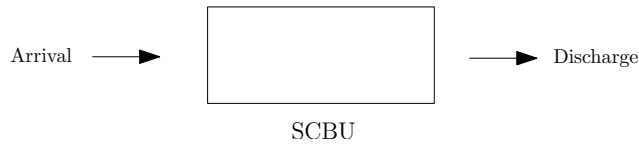

Arrival

SCBU

Discharge

Figure 2: Sub-network model for a level 1 unit.

equipment resources, and will be referred to as 'overflow'. Rejection occurs only when all cots are occupied; in such cases neonates are transferred to another neonatal unit. Patient flows in a typical level 3 or level 2 unit are depicted in Figure 4. Unlike for level 3/level 2 units, overflow does not occur in level 1 units with ITU.

The underlying admission, discharge and transfer policies of a perinatal network are described below.

1. All mothers expecting birth < 27 week of gestational age or all neonates
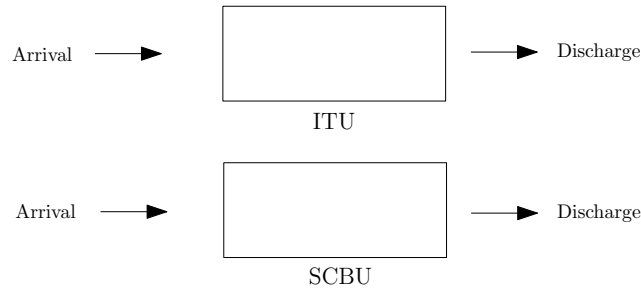
Figure 3: Sub-network model for a level 1 unit with ITU.
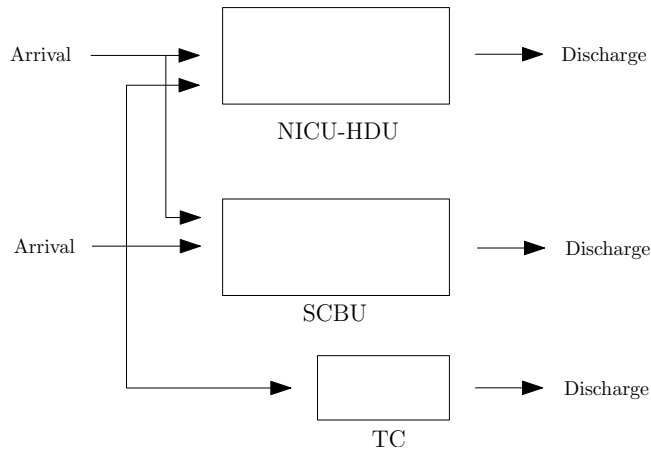
Figure 4: Sub-network model for a level 3/level 2 unit.

with < 27 week of gestational age are transferred to a level 3 unit.

2. All mothers expecting birth ≥ 27 but < 34 week of gestational age or all neonates of the same gestational age are transferred to a level 2 unit depending upon the booked place of delivery.

3. All neonatal units accept neonates for special care booked at the same unit.

4. Neonates admitted into units other than their booked place of delivery are transferred back to their respective neonatal unit receiving after the required level of care.

Now we shall develop a generalised loss network framework for a perinatal network with level 1, level 2 and level 3 units. To obtain the steady state behavior of the network, we first decompose the whole network into a set of subnetworks (i.e., each neonatal units) due to higher dimensionality, then we derive the steady state solution and expression of rejection probability for each of the units. When analysing a particular sub-network in isolation, back transfers are combined with new arrivals to specifically take into account the dependencies between units. Cot capacity for the neonatal units may be determined based on the rejection probabilities at each level of care and overflow to temporary care of the units.

3. Mathematical model formulation

3.1. Model for a level 1 unit

A level 1 unit consists typically of a SCBU. Therefore, assuming no waiting space and first come first served (FCFS) discipline, a level 1 unit can be modelled as a GI/G/c/0 system. Let the inter-arrival times and length of stay of neonates be i.i.d. random variables denoted by A and L, respectively. Also the length of stay is independent of the arrival process. Define

$$m_A = \mathbb{E}(A) = \frac{1}{\lambda}, \qquad\qquad m_L = \mathbb{E}(L) = \frac{1}{\mu}.$$

Let N denotes the number of neonates in the system at an arbitrary time, N^a denotes the number of neonates (arriving) who find the system is in steady state with N neonates, and N^d denotes the number of neonates discharged from the system in steady state with N neonates. Let c be the number of cots at the SCBU. For $0 \leq n \leq c$, let

$$\pi(n) = \mathbb{P}(N = n),$$

$$\pi^a(n) = \mathbb{P}(N^a = n), \qquad\qquad \pi^d(n) = \mathbb{P}(N^d = n),$$

and

$$m_{A,n}^d = \mathbb{E}(A_n^d), \qquad m_{L,n}^a = \mathbb{E}(L_n^a), \qquad m_{L,n}^d = \mathbb{E}(L_n^d),$$

where A_n^d is the remaining inter-arrival time at the discharge instant of a neonate who leaves behind n neonates in the systems, L_n^a (L_n^d) is the remaining length of stay of a randomly chosen occupied cot at the arrival (discharge) instant of a neonate who finds (leaves behind) n neonates in the system.

Let $m_{A,n}^a$ and $m_{L,n}^{*a}$ be, respectively, the mean inter-arrival time and the mean length of stay under the condition that the system started at the arrival instant of a neonate when there were n neonates in the system. Clearly,

$$m_{A,n}^a = m_A, \qquad\qquad m_{L,n}^{*a} = m_L\,.$$

From the definitions, we obtain

$$m_{A,c}^d = m_A, \qquad\qquad m_{L,c}^a = m_{L,c}^d\,.$$

We set

$$m_{A,-1}^a = 0, \qquad m_{A,-1}^d = 0, \qquad m_{L,0}^a = 0, \qquad m_{L,0}^d = 0,$$

for convenience. Then the first set of system equations obtained by Franken et al. (1982) for a GI/G/c/0 loss system can be written as

$$\pi(n) - \lambda m_{A,n-1}^a \pi^a(n-1) = -\lambda m_{A,n-1}^d \pi^d(n-1) + \lambda m_{A,n}^d \pi^d(n), \ \ 0 \le n \le c.$$

The second set of system equations can be given by

$$n\pi(n) + (n-1)\lambda m_{L,n-1}^d \pi^d(n-1) - n\lambda m_{L,n}^d \pi^d(n)$$

$$= \lambda m_{L,n-1}^{*a} \pi^a(n-1) + (n-1)\lambda m_{L,n-1}^a \pi^a(n-1) - n\lambda m_{L,n}^a \pi^a(n), \quad 1 \le n \le c-1,$$

and

$$c\pi(c) + (c-1)\lambda m_{L,c-1}^d \pi^d(c-1) = \lambda m_{L,n}^{*a} \pi^a(c-1) + (c-1)\lambda m_{L,c-1}^a \pi^a(c-1).$$

From the first set of system equations for the GI/G/c/0 queue, the following equations can be derived

$$\pi(0) = \lambda m_{A,0}^d \pi^d(0), \tag{1}$$

and

$$\pi(n) = \lambda m_{A,n}^d \pi^d(n) + \lambda m_A \pi^a(n-1) - \lambda m_{A,n-1}^d \pi^d(n-1), \quad 1 \le n \le c. \tag{2}$$

From the second set of system equations for the GI/G/c/0 queue, the following equations can be derived

$$\pi(n) = \frac{1}{n}\Big[\lambda \pi^a(n-1)\big(m_L + (n-1)m_{L,n-1}^a\big) + \lambda n m_{L,n}^d \pi^d(n)$$

$$- (n-1)\lambda m_{L,n-1}^d \pi^d(n-1) - \lambda n m_{L,n}^a \pi^a(n)\Big], \quad 1 \le n \le c-1, \tag{3}$$

and

$$\pi(c) = \frac{1}{c}\Big[\lambda \pi^a(c-1)\big(m_L + (c-1)m_{L,c-1}^a\big) - (c-1)\lambda m_{L,c-1}^d \pi^d(c-1)\Big]. \tag{4}$$

Theorem 1. *The steady state distribution for a GI/G/c/0 system is given by*

$$\pi^a(n) = \pi^d(n) = K^{-1} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}, \quad 0 \le n \le c, \tag{5}$$

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and

$$\pi(n) = \pi^a(n)\varphi_n = \pi^d(n)\varphi_n, \ \ 0 \le n \le c,$$

where

$$K = \sum_{n=0}^{c} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}},$$

and

$$
\left.
\begin{aligned}
\frac{1}{\mu_i} &= m_L - i\big(m_A - m_{A,i-1}^d\big) + (i-1)\big(m_{L,i-1}^a - m_{L,i-1}^d\big), \ \ 1 \le i \le c, \\[1mm]
\frac{1}{\lambda_i} &=
\begin{cases}
(i+1)\big(m_{A,i+1}^d + m_{L,i+1}^a - m_{L,i+1}^d\big), \ \ 0 \le i \le c-2, \\[1mm]
c m_A, \ \ i = c-1,
\end{cases} \\[2mm]
\varphi_i &=
\begin{cases}
\lambda m_{A,0}^d, \ \ i = 0, \\[1mm]
\lambda\Big[m_{A,i}^d + \big(m_A - m_{A,i-1}^d\big)\mu_i/\lambda_{i-1}\Big], \ \ 1 \le i \le c.
\end{cases}
\end{aligned}
\right\}
$$

$$(6)$$

Proof. The steady state distribution can be obtained by solving the above two sets of system equations. First, we equate equations (1) and (2) with equations (3) and (4) for each n, $0 \le n \le c$. Then using the following well-known rate conservation principle, we solve them simultaneously,

$$\pi^a(n) = \pi^d(n).$$

Hence we obtain equation (5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

In steady state analysis of a GI/G/c/0 system, equations in (6) involve quantities $m_{A,n}^d$, $m_{L,n}^a$ and $m_{L,n}^d$, which are not easy to compute in general, except for some special cases such as Poisson arrival or exponential length of stay. Therefore, a two moment approximation is used as proposed by Kim and Chae (2003) and Choi et al. (2005) for the steady state distribution of the

GI/G/c/0 system based on the exact results as derived in equations 5 and 6. To obtain the approximation, we replace the inter-arrival and length of stay average quantities $m^d_{A,n}$, $m^a_{L,n}$ and $m^d_{L,n}$ by their corresponding time-average quantities;

$$m^d_{A,n} \approx q_A = \frac{\mathbb{E}(A^2)}{2\mathbb{E}(A)} = \frac{(1+c_A^2)m_A}{2}, \quad 0 \le n \le c-1, \tag{7}$$

$$m^a_{L,n} = m^d_{L,n} \approx q_L = \frac{\mathbb{E}(L^2)}{2\mathbb{E}(L)} = \frac{(1+c_L^2)m_L}{2}, \quad 0 \le n \le c-1, \tag{8}$$

where c_A^2 (c_L^2) is the squared coefficient of variation of inter-arrival times (length of stay).

Using equations (7) and (8) in equation (5), we obtain the two moment approximation for the steady state distribution

$$\tilde{\pi}^a(n) = \tilde{\pi}^d(n) = \tilde{K}^{-1} \prod_{i=0}^{n-1} \frac{\tilde{\lambda}_i}{\tilde{\mu}_{i+1}}, \quad 0 \le n \le c, \tag{9}$$

and

$$\tilde{\pi}(n) = \tilde{\pi}^a(n)\tilde{\varphi}_n = \tilde{\pi}^d(n)\tilde{\varphi}_n, \quad 0 \le n \le c,$$

where

$$\tilde{K} = \sum_{n=0}^{c} \prod_{i=0}^{n-1} \frac{\tilde{\lambda}_i}{\tilde{\mu}_{i+1}},$$

and

$$\left. \begin{array}{l} \dfrac{1}{\tilde{\mu}_i} = m_L - i(m_A - q_A), \quad 1 \le i \le c, \\[2ex] \dfrac{1}{\tilde{\lambda}_i} = \begin{cases} (i+1)q_A, \ 0 \le i \le c-2, \\[1ex] cm_A, \ i = c-1, \end{cases} \\[3ex] \tilde{\varphi}_i = \begin{cases} \lambda q_A, \ i = 0, \\[1ex] \lambda\left[q_A + (m_A - q_A)\tilde{\mu}_i/\tilde{\lambda}_{i-1}\right], \ 1 \le i \le c-1, \\[1ex] \lambda\left[m_A + (m_A - q_A)\tilde{\mu}_i/\tilde{\lambda}_{i-1}\right], \ i = c. \end{cases} \end{array} \right\} \tag{10}$$

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Therefore, the rejection probability for a level 1 unit is computed as

$$R = \tilde{\pi}(n) \Big/ \sum_{n=0}^{c} \tilde{\pi}(n).$$

3.2. Model for a level 1 neonatal unit with ITU

In a level 1 unit with ITU (Figure 3), overflow from ITU to SCBU does not occur. The unit can be modelled as two joint GI/G/c/0 systems. Therefore, extending the Theorem 1, the steady state distribution for a level 1 neonatal unit with ITU is given by

$$\pi^a(\mathbf{n}) = \pi^d(\mathbf{n}) = K^{-1} \prod_{i=0}^{(n_1-1)} \prod_{j=0}^{(n_2-1)} \frac{\lambda_{1i}}{\mu_{1(i+1)}} \cdot \frac{\lambda_{2j}}{\mu_{2(j+1)}},$$

and

$$\pi(\mathbf{n}) = \pi^a(\mathbf{n})\varphi_{\mathbf{n}},$$

where

$$K = \sum_{n_1,n_2} \prod_{i=0}^{(n_1-1)} \prod_{j=0}^{(n_2-1)} \frac{\lambda_{1i}}{\mu_{1(i+1)}} \cdot \frac{\lambda_{2j}}{\mu_{2(j+1)}}.$$

The approximate steady state distribution for a level 1 neonatal unit with ITU is given by

$$\tilde{\pi}^a(\mathbf{n}) = \tilde{\pi}^d(\mathbf{n}) = \tilde{K}^{-1} \prod_{i=0}^{(n_1-1)} \prod_{j=0}^{(n_2-1)} \frac{\tilde{\lambda}_{1i}}{\tilde{\mu}_{1(i+1)}} \cdot \frac{\tilde{\lambda}_{2j}}{\tilde{\mu}_{2(j+1)}},$$

and

$$\tilde{\pi}(\mathbf{n}) = \tilde{\pi}^a(\mathbf{n})\tilde{\varphi}_{\mathbf{n}},$$

where

$$\tilde{K} = \sum_{n_1,n_2} \prod_{i=0}^{(n_1-1)} \prod_{j=0}^{(n_2-1)} \frac{\tilde{\lambda}_{1i}}{\tilde{\mu}_{1(i+1)}} \cdot \frac{\tilde{\lambda}_{2j}}{\tilde{\mu}_{2(j+1)}}.$$

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and $\tilde{\lambda}_{1i}$, $\tilde{\mu}_{1i}$, $\tilde{\lambda}_{2i}$, $\tilde{\mu}_{2i}$ and $\tilde{\varphi}_i$ are defined by equations in (10) for NICU-HDU and SCBU-TC, respectively.

The rejection probability at the ith level of care is calculated as

$$R_i = \sum_{\mathbf{n} \in T_i} \tilde{\pi}(\mathbf{n}) \Big/ \sum_{\mathbf{n} \in S} \tilde{\pi}(\mathbf{n}), \quad i = 1, 2,$$

where $T_1 = \big\{\mathbf{n} \in S \mid n_1 = c_1\big\}$ and $T_2 = \big\{\mathbf{n} \in S \mid n_2 = c_2\big\}$.

3.3. Model for a level 3/level 2 neonatal unit

We derive the mathematical model for a level 3/level 2 neonatal unit as described in Section 2 and showing in Figure 4. Let c_1, c_2 and c_3 be the number of cots at NICU-HDU, SCBU and TC, respectively. Let $X_i(t)$ be the number of neonates at unit i, and $X_{ij}(t)$ be the number of neonates overflowing from unit i to unit j, $i, j \in \{1, 2, 3\}$ at time t. Then the vector process

$$\mathbf{X} = \big(X_1(t), X_{12}(t), X_2(t), X_{21}(t), X_{23}(t), t \geq 0\big)$$

is a continuous-time discrete-valued stochastic process. We assume the process is time homogeneous, aperiodic and irreducible on its finite state space. The process does not necessarily need to hold the Markov property. The state space is given by

$$S = \big\{\mathbf{n} = (n_1, o_{12}, n_2, o_{21}, o_{23}) : n_1 + o_{21} \leq c_1, o_{12} + n_2 \leq c_2, o_{23} \leq c_3\big\},$$

where $n_i, i = 1, 2$, is the number of neonates at the ith main unit, and $o_{ij}, i, j \in \{1, 2, 3\}$, is the number of neonates at the jth overflow unit from the ith unit. Now the system can be modelled as two joint loss queueing processes with overflow. Assume that the joint GI/G/c/0 systems are in steady state.

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We shall now derive the expression for the steady state distribution for a level 3/level 2 neonatal unit. Extending the Theorem 1 for two joint GI/G/c/0 systems, the steady state distribution for a level 3 or level 2 neonatal unit with overflows can be derived.

Theorem 2. *The steady state distribution for a level 3 or level 2 unit can be given by*

$$\pi^a(\mathbf{n}) = \pi^d(\mathbf{n}) = K^{-1} \prod_{i=0}^{(n_1+o_{21}-1)} \prod_{j=0}^{(n_2+o_{12}+o_{23}-1)} \frac{\lambda_{1i}}{\mu_{1(i+1)}} \cdot \frac{\lambda_{2j}}{\mu_{2(j+1)}},$$

and

$$\pi(\mathbf{n}) = \pi^a(\mathbf{n})\varphi_{\mathbf{n}},$$

where λ_{1i}, λ_{2j}, μ_{1i}, μ_{2j}, φ_i are arrival and departure related quantities for NICU-HDU and SCBU-TC, respectively, defined by equations in (6), and

$$K = \sum_{\mathbf{n}\in S} \prod_{i=0}^{(n_1+o_{21}-1)} \prod_{j=0}^{(n_2+o_{12}+o_{23}-1)} \frac{\lambda_{1i}}{\mu_{1(i+1)}} \cdot \frac{\lambda_{2j}}{\mu_{2(j+1)}}$$

is the normalising constant.

The approximate steady state distribution for a level 3/level 2 neonatal unit is given by

$$\tilde{\pi}^a(\mathbf{n}) = \tilde{\pi}^d(\mathbf{n}) = \tilde{K}^{-1} \prod_{i=0}^{(n_1+o_{21}-1)} \prod_{j=0}^{(n_2+o_{12}+o_{23}-1)} \frac{\tilde{\lambda}_{1i}}{\tilde{\mu}_{1(i+1)}} \cdot \frac{\tilde{\lambda}_{2j}}{\tilde{\mu}_{2(j+1)}},$$

and

$$\tilde{\pi}(\mathbf{n}) = \tilde{\pi}^a(\mathbf{n})\tilde{\varphi}_{\mathbf{n}},$$

where $\tilde{\lambda}_{1i}$, $\tilde{\mu}_{1i}$, $\tilde{\lambda}_{2i}$, $\tilde{\mu}_{2i}$ and $\tilde{\varphi}_i$ are defined by equations in (10) for NICU-HDU and SCBU-TC, respectively, and

$$\tilde{K} = \sum_{\mathbf{n}\in \mathbf{S}} \prod_{i=0}^{(n_1+o_{21}-1)} \prod_{j=0}^{(n_2+o_{12}+o_{23}-1)} \frac{\tilde{\lambda}_{1i}}{\tilde{\mu}_{1(i+1)}} \cdot \frac{\tilde{\lambda}_{2j}}{\tilde{\mu}_{2(j+1)}}.$$

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The rejection probability at the ith level of care for a level 3/level 2 neonatal unit is computed as

$$R_i = \sum_{\mathbf{n} \in T_i} \tilde{\pi}(\mathbf{n}) \Big/ \sum_{\mathbf{n} \in S} \tilde{\pi}(\mathbf{n}), \tag{11}$$

where

$$T_1 = \big\{ \mathbf{n} \in S \mid (n_1 + o_{21} = c_1 \ \text{and} \ o_{12} + n_2 = c_2) \big\},$$

and

$$T_2 = \big\{ \mathbf{n} \in S \mid (o_{12} + n_2 = c_2, \ n_1 + o_{21} = c_1 \ \text{and} \ o_{23} = c_3) \big\}.$$

The overflow probability $O_i, i = 1, 2$ at the ith level of care for a level 3/level 2 unit can also be computed from equation (11) substituting T_i by $\{T_i^* \setminus T_i\}, i = 1, 2$,

where

$$T_1^* = \big\{ \mathbf{n} \in S \mid (n_1 = c_1 \ \text{and} \ o_{12} + n_2 < c_2) \big\},$$

and

$$T_2^* = \big\{ \mathbf{n} \in S \mid (n_2 + o_{12} = c_2 \ \text{and} \ n_1 + o_{21} < c_1) \ \text{or}$$
$$(o_{12} + n_2 = c_2, n_1 + o_{21} = c_1 \ \text{and} \ o_{23} < c_3) \big\}.$$

Corollary 3. *The approximate steady state distribution for a level 3 or level 2 neonatal unit is exact for exponential inter-arrival time and length of stay distributions at each level of care.*

Proof. In the case of exponential inter-arrival time and length of stay distributions, arrival and departure related parameters reduce to the corresponding mean values of inter-arrival and length of stay

$$m_{1A,n}^d = q_{1A} = m_{1A} = \frac{1}{\lambda_1}, \qquad m_{1L,n}^a = m_{1L,n}^d = q_{1L} = m_{1L} = \frac{1}{\mu_1}$$

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$$m_{2A,n}^d = q_{2A} = m_{2A} = \frac{1}{\lambda_2}, \qquad m_{2L,n}^a = m_{2L,n}^d = q_{2L} = m_{2L} = \frac{1}{\mu_2}$$

and

$$\varphi_{\mathbf{n}} = 1.$$

Then the steady state solution becomes

$$\pi^a(\mathbf{n}) = \pi^d(\mathbf{n}) = K^{-1} \prod_{i=0}^{(n_1+o_{21}-1)} \frac{\lambda_1}{(i+1)\mu_1} \prod_{j=0}^{(n_2+o_{12}+o_{23}-1)} \frac{\lambda_2}{(j+1)\mu_2}.$$

Hence we obtain

$$\pi(\mathbf{n}) = K^{-1} \frac{\left(\frac{\lambda_1}{\mu_1}\right)^{(n_1+o_{21})} \left(\frac{\lambda_2}{\mu_2}\right)^{(o_{12}+n_2+o_{23})}}{(n_1+o_{21})!(o_{12}+n_2+o_{23})!},$$

where

$$K = \sum_{\mathbf{n} \in S} \frac{\left(\frac{\lambda_1}{\mu_1}\right)^{(n_1+o_{21})} \left(\frac{\lambda_2}{\mu_2}\right)^{(o_{12}+n_2+o_{23})}}{(n_1+o_{21})!(o_{12}+n_2+o_{23})!},$$

which is the steady state solution for a level 3 unit as in Asaduzzaman and Chaussalet (2011) for Markovian arrival and discharge patterns. Adding back transfers, we can easily obtain the steady state distribution for a level 2 unit. □

4. Application of the model

4.1. The case study

We apply the model to the case of a perinatal network in London which is the north central London perinatal network (NCLPN). The network consists of five neonatal units: UCLH (level 3), Barnet (level 2), Whittington (level 2), Royal Free (level 1 with ITU) and Chase Farm (level 1). The underlying

Table 1: Inter-arrival and length of stay for the neonatal units in the NCLPN in 2008

Unit	Mean inter-arrival	Mean length of stay
UCLH		
NICU-HDU	0.58	11.51
SCBU-TC	0.24	5.83
Barnet		
NICU-HDU	1.12	6.78
SCBU-TC	0.83	9.71
Whittington		
NICU-HDU	1.11	5.16
SCBU-TC	0.88	14.61
Royal Free		
ITU	2.77	2.21
SCBU	0.91	9.99
Chase Farm		
SCBU	1.05	8.03

aim of the network is to achieve capacity so that 95% women and neonates may be cared for within the network.

Data on admission and length of stay were provided by each of the units. Since the data did not contain the actual arrival rate and the rejection probability for the units we estimated the actual arrival rates using SIMUL8® (SIMUL8, 2000), a computer simulation package designed to model and measure performance of a stochastic service system. Table 1 presents mean length of stay and estimated mean inter-arrival times for each level of care at UCLH, Barnet, Whittington, Royal Free and Chase Farm neonatal units for the year

2008. Then we also use simulation (SIMUL8) to estimate the rejection probabilities for each level of care of the units for various arrival and discharge patterns. We refer to these estimates as 'observed' rejection probabilities.

4.2. Numerical results and discussion

In this section rejection probabilities are estimated for all five units in the NCLPN through the application of the model formulae in Section 3. An extensive numerical investigation has been carried out for a variety of inter-arrival and length of stay distributions to test the performance of the model and the approximation method. A common approach to check the accuracy of a generalised queueing model is to consider a variety of interarrival and LOS distributions and compare performance measures derived through the model with exact or simulated results (Chandy et al., 1975; Kerbachea and Smith, 1987; Altiok, 1989; Smith, 2003; Choi et al., 2005; Atkinson, 2009). In this study, exponential (M), two-phase hyper-exponential (H_2) and two-phase Erlang (E_2) distributions are considered to derive rejection probabilities as these were suggested to be appropriate interarrival and LOS distributions for the five neonatal hospitals we considered (Asaduzzaman et al., 2010a; Asaduzzaman, 2010). Hyper-exponential distributions were also found apprpriate by Griffiths et al. (2006) in the context of a classical ICU.

Table 2 compares the 'observed' and estimated rejection probabilities at each level of care for UCLH, Barnet, Whittington, Royal Free and Chase Farm neonatal units for various combinations of inter-arrival time and length of stay distributions. Namely, exponential (M), two-phase hyper-exponential (H_2) and two-phase Erlang (E_2) distributions are considered. To compare 'observed' rejection probabilities with estimated rejection probabilities when

Table 2: Comparison of rejection probabilities for different distributions at all five neonatal units in the NCLPN

UCLH (17 NICU, 12 SCBU and 8 TC cots)	System notation	'Observed' rej. prob.	Est. rej. prob.	Abs. per. err.
NICU-HDU	$M/M/c/0$	0.1895	0.1962	3.54
SCBU-TC		0.1319	0.1271	3.64
NICU-HDU	$M/H_2/c/0$	0.1989	0.1933	2.82
SCBU-TC		0.1186	0.1313	10.71
NICU-HDU	$H_2/M/c/0$	0.2123	0.1706	19.64
SCBU-TC		0.1214	0.1010	16.80
NICU-HDU	$M/E_2/c/0$	0.2096	0.1987	5.20
SCBU-TC		0.1405	0.1235	12.10
NICU-HDU	$E_2/M/c/0$	0.2179	0.2347	7.71
SCBU-TC		0.1332	0.1652	24.02
NICU-HDU	$H_2/H_2/c/0$	0.1852	0.1669	9.88
SCBU-TC		0.1255	0.1077	14.18
NICU-HDU	$H_2/E_2/c/0$	0.1848	0.1726	4.98
SCBU-TC		0.0996	0.0970	2.61
NICU-HDU	$E_2/H_2/c/0$	0.2155	0.2332	8.21
SCBU-TC		0.1512	0.1672	10.58
NICU-HDU	$E_2/E_2/c/0$	0.2260	0.2367	4.73
SCBU-TC		0.1353	0.1626	20.18
Barnet (6 NICU, 14 SCBU and 4 TC cots)				
NICU-HDU	$M/M/c/0$	0.1644	0.1508	8.27
SCBU-TC		0.0142	0.0076	*
NICU-HDU	$M/H_2/c/0$	0.1496	0.1614	7.89
SCBU-TC		0.0117	0.0111	*
NICU-HDU	$H_2/M/c/0$	0.1411	0.1513	7.23
SCBU-TC		0.0147	0.0097	*
NICU-HDU	$M/E_2/c/0$	0.1653	0.1433	13.31
SCBU-TC		0.0141	0.0051	*
NICU-HDU	$E_2/M/c/0$	0.1326	0.1529	15.31
SCBU-TC		0.0055	0.0020	*
NICU-HDU	$H_2/H_2/c/0$	0.1586	0.1571	0.95
SCBU-TC		0.0125	0.0134	*
NICU-HDU	$H_2/E_2/c/0$	0.1508	0.1473	2.32
SCBU-TC		0.0142	0.0072	*
NICU-HDU	$E_2/H_2/c/0$	0.1691	0.1752	3.61
SCBU-TC		0.0034	0.0037	*
NICU-HDU	$E_2/E_2/c/0$	0.1269	0.1355	6.78
SCBU-TC		0.0059	0.0007	*

*APEs are ignored for rejection probabilities < 0.05

Continuation of Table 2

Whittington (12 NICU, 16 SCBU and 5 TC cots)	System notation	'Observed' rej. prob.	Est. rej. prob.	Abs. per. err.
NICU-HDU	$M/M/c/0$	0.0216	0.0007	*
SCBU-TC		0.0138	0.0018	*
NICU-HDU	$M/H_2/c/0$	0.0009	0.0026	*
SCBU-TC		0.0003	0.0128	*
NICU-HDU	$H_2/M/c/0$	0.0042	0.0000	*
SCBU-TC		0.0110	0.0011	*
NICU-HDU	$M/E_2/c/0$	0.0097	0.0015	*
SCBU-TC		0.0029	0.0054	*
NICU-HDU	$E_2/M/c/0$	0.0006	0.0000	*
SCBU-TC		0.0010	0.0011	*
NICU-HDU	$H_2/H_2/c/0$	0.0053	0.0035	*
SCBU-TC		0.0091	0.0225	*
NICU-HDU	$H_2/E_2/c/0$	0.0002	0.0026	*
SCBU-TC		0.0236	0.0134	*
NICU-HDU	$E_2/H_2/c/0$	0.0003	0.0000	*
SCBU-TC		0.0002	0.0024	*
NICU-HDU	$E_2/E_2/c/0$	0.0018	0.0000	*
SCBU-TC		0.0005	0.0005	*
Royal Free (2 ITU and 12 SCBU cots)				
ITU	$M/M/c/0$	0.1468	0.1504	2.45
SCBU		0.1558	0.1580	1.41
ITU	$M/H_2/c/0$	0.1714	0.1504	12.25
SCBU		0.1476	0.1580	7.05
ITU	$H_2/M/c/0$	0.1667	0.1556	6.66
SCBU		0.1509	0.1476	2.19
ITU	$M/E_2/c/0$	0.1560	0.1504	3.59
SCBU		0.1393	0.1580	13.42
ITU	$E_2/M/c/0$	0.1756	0.1504	14.35
SCBU		0.1516	0.1685	11.15
ITU	$H_2/H_2/c/0$	0.1681	0.1351	19.63
SCBU		0.1452	0.1476	1.65
ITU	$H_2/E_2/c/0$	0.1481	0.1556	5.06
SCBU		0.1680	0.1476	12.14
ITU	$E_2/H_2/c/0$	0.1252	0.1347	7.59
SCBU		0.1384	0.1685	21.75
ITU	$E_2/E_2/c/0$	0.1315	0.1579	20.08
SCBU		0.1619	0.1685	4.08

Continuation of Table 2

Chase Farm (10 SCBU cots)	System notation	'Observed' rej. prob.	Est. rej. prob.	Abs. per. err.
SCBU	M/M/c/0	0.1078	0.1060	1.67
SCBU	M/H_2/c/0	0.1094	0.1060	3.11
SCBU	H_2/M/c/0	0.1474	0.1233	16.35
SCBU	M/E_2/c/0	0.1047	0.1060	1.24
SCBU	E_2/M/c/0	0.0719	0.0792	10.15
SCBU	H_2/H_2/c/0	0.1418	0.1233	13.0
SCBU	H_2/E_2/c/0	0.1469	0.1233	16.0
SCBU	E_2/H_2/c/0	0.0817	0.0792	3.06
SCBU	E_2/E_2/c/0	0.0700	0.0792	13.14

one of these probabilities are 0.05 or more, we define 'absolute percentage error' (APE) as the absolute deviation between 'observed' and estimated rejection probability divided by 'observed' rejection probability and then multiplied by 100. Rejection probabilities below 0.05 are normally considered satisfactory. For this reason we have not reported the APE when both 'observed' and estimated rejection probabilities are less than 0.05.

The 'observed' and estimated rejection probabilities are close for the UCLH unit. At NICU-HDU, the highest 'observed' rejection probability is occurred for E_2/E_2/c/0, and the estimated rejected probability is also highest for the same arrival and discharge patterns with an absolute percentage error (APE) 4.73%. The lowest 'observed' rejection probability is 0.1848 for the H_2/E_2/c/0 while the estimated rejection probability is 0.1726 with an APE 4.98%. At SCBU for E_2/M/c/0, the 'observed' and estimated rejection probabilities are 0.1332 and 0.1652, respectively, with an APE 24.02%. At Barnet NICU-HDU, the 'observed' and estimated rejection probabilities are close with a varying APEs from 0.95%–15.31%. For Barnet SCBU the 'observed' and estimated rejection probabilities are all less than 0.05 and

relatively close to each other. Both the UCLH NICU-HDU and SCBU and Barnet NICU-HDU would require additional cots to keep the rejection level low and achieve a 0.05 target.

Rejection probabilities from both NICU-HDU and SCBU at the Whittington neonatal unit are below 0.05 regardless of the combination of inter-arrival time and length of stay distributions, which indicates that the neonatal unit is performing well with 12 NICU, 16 SCBU and 5 TC cots. The 'observed' and estimated rejection probabilities at Royal Free ITU and SCBU and Chase Farm SCBU are close to each other. The results in Table 2 suggest that Royal Free ITU and SCBU and Chase Farm SCBU require extra cots to decrease the rejection level.

Through our extensive numerical investigations we observe that the rejection probability often varies greatly according to arrival and discharge patterns. The number of cots required will also vary depending upon arrival and discharge patterns. Therefore, one should take into account the actual arrival and discharge patterns for accurate capacity planning of neonatal units rather than approximating by Markovian arrival and discharge patterns. To achieve a '95%' admission acceptance target UCLH NICU-HDU and SCBU, Barnet NICU-HDU, Royal Free ITU and SCBU, and Chase Farm SCBU need to increase their number of cots.

We have also observed that performance of the proposed generalised capacity planning model improves as the squared coefficient of variation values of inter-arrival and length of stay get closer to 1 (recall that our approximation is exact for the Markovian inter-arrival and length of stay case in which squared coefficient of variation values of inter-arrival and length of stay are

both 1) and as λ/μ gets larger (i.e., under heavy traffic). A possible explanation is that as λ/μ gets larger, the period during which all the cots are busy tends to get longer. As such a busy period gets longer, arrival and departure points of arrivals tend to become more and more like arbitrary points in time. As such, the approximation is likely to get more accurate.

5. Conclusion

Planning capacity accurately has been an important issue in the neonatal sector because of the high cost of care, in particular. Markovian arrival and length of stay can provide only approximate estimates which may often underestimate or overestimate the required capacity. The underestimation of cots may increase the rejection level, which in turn may be life-threatening or cause expensive transfers for high risk neonates, hence increase risk for vulnerable babies. On the other hand, overestimation may cause under-utilisation of cots, and potential waste of resources.

In this paper a generalised framework for determining cot capacity of a perinatal network was derived. After decomposing the whole network into neonatal units, each unit was analysed separately. Expressions for the stationary distribution and for rejection probabilities were derived for each neonatal unit. An approximation method was suggested to obtain the steady state rejection probabilities. The model formulation was then applied to the neonatal units in the NCLPN. A variety of inter-arrival and length of stay distributions in the neonatal units has been considered for numerical experimentation. The 'observed' and estimated rejection probabilities were close (APE typically less than 20%) for all hospital units when rejection proba-

bilities were 0.05 or more. When 'observed' rejection probabilities were less than 0.05, as for the Barnet SCBU and both the Whittington NICU-HDU and SCBU, the APE increased rapidly to beyond 50%. However, since these values are less than or close to 0.05, they do not have an impact on management decisions regarding the number of cots. In contrast, when 'observed' rejection probabilities are high, then the estimated values become close to each other. The 'observed' and estimated rejection probabilities were, in general, close for high traffic intensities. As traffic intensity drops the absolute percent error increases quickly. In most cases, the absolute percent error becomes small for Markovian arrival and length of stay patterns. We know that service time distribution is insensitive for delay systems if the arrival process is Poisson. However, the property is no longer valid for loss systems. The model results as seen in Table 2 also confirm this sensitivity property.

The main advantage of the model framework is that arrival and discharge pattern do not need to hold the Markov property. The model is based on the first two moments and requires no distributional assumption. This two-moment approximation techniques performs reasonably well in terms of accuracy (APE) and is fast. The method is exactly Markovian for equal mean and variance. The numerical results show that the model can be used as a capacity planning tool for perinatal networks for non-Markovian arrival and discharge patterns as well as Markovian patterns. If good estimates of the first two moments are available, then the generalised model can be used to determine the required cot capacity in a perinatal network for given level of rejection probabilities.

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