The Further Education of Gifted Mathematicians: A Vygotskian Perspective on Perceptions and Pedagogy.

Niall Thompson

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Dedication and Acknowledgements

Dedication

For Kate, in the sincere hope that nineteen years of the joy of maths, and five years working on this research while teaching other gifted mathematicians, will mean I one day become half the teacher, mentor, and friend you know so naturally how to be.

For Iris, whose pride I can feel today as keenly as the day I became a student half my lifetime ago. It took a while to find my way back to academia Grandma, but we got there.

For Cliff. I do not recall a time I did not know my times tables, and yet I remember it was you who insisted I learn them and grilled me on them daily. Thank you, Grandad, for my earliest memories of numbers. My hardest-earned certificate is destined for your wall.

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Abstract

Gifted mathematicians often experience challenges developing as mathematicians for the first time during their further education, an educational phase in which national policy prioritises the study of mathematics for lower-attaining students. Institutions do not therefore routinely provide the specialised support gifted mathematicians require to overcome the challenges associated with this phase. To investigate the nature of their challenges and develop a pedagogical model to support their advanced mathematical-development, this study invited three Year 12 gifted mathematicians from an English 16-19 free school to participate in advanced problem-solving interventions. They kept digital-diaries for four weeks, and participated in interviews to reflect on their experiences. Interpretative phenomenological analysis was refined to facilitate a chronological analysis of their data, which established a detailed picture of their respective successes and challenges. Their views were then critically evaluated collectively, to create a joint understanding of their support needs. The Vygotskian theoretical perspective was honed for application as a theoretical framework to explore the nuances of their perceptions. This facilitated an improved understanding of: adjustment to the abstract nature of problem-solving required throughout advanced mathematical-development; the process through which gifted mathematicians can utilise their feelings of frustration to fuel their motivation to nonetheless continue developing; the strategies through which practitioners can effectively scaffold this development; and, subsequently, how gifted mathematicians might situate themselves within the social context of advanced mathematical-development to facilitate their own success. An effective balance between a gifted mathematician’s need to make tangible progress and their competing need to perceive the role of their own independence as a significant factor when making that progress was found to be a core consideration which evolved as they developed. The study establishes a first pedagogical model of advanced mathematical-development for supporting a gifted mathematician to make progress with a growing sense of independent capability.

Glossary

Definitions of Key Terms

Term (Cross-Reference)	Definition
Advanced Mathematical-Development	The pursuit of knowledge and skills beyond those listed in the A-Level Mathematics and Further Mathematics specifications, but that nonetheless takes place during the further education phase.
Appropriate Challenge (2.4.2)	A task with a difficulty level inside a person's Zone of Proximal Development. This therefore varies over time, and between people.
AS/A-Level	Advanced/Advanced Subsidiary Levels are qualifications ordinarily taken by learners on academic programmes throughout and at the conclusion of their further education phase. AS-Levels may be taken by students in Year 12 or Year 13. A-Levels build on AS-Levels by extending topics and introducing additional topics. The A-Level is approximately double the length of the AS-Level and is usually only taken in Year 13.
Developable Talent (2.4.1)	A talent which has the potential to arise or has arisen specifically through the application of sustained effort and focused support.
Free School	A school which is funded by the Department for Education but that is not maintained by a local authority, and so can create its own curriculum.
Further Education	Further Education (FE, capitalised) is a sector in the English education system comprised of institutions which educate learners of ages 14-19. It is predominantly comprised of FE colleges, which ordinarily focus on vocational programmes. However, the sector also includes sixth form colleges which deliver A-Level programmes, and niche A-Level settings such as the specialist mathematics school where the research took place.
further education	The term 'further education' (uncapitalised) refers to the phase in an individual's education which is post-GCSE and pre-university

	and is typically undertaken from ages 16-19 in the English education system. In addition to being undertaken at FE institutions, the ‘further education’ phase is also undertaken at 11-18 schools by learners in Year 12 and Year 13.
Gifted Learner	A person with a quality of giftedness in some subject or domain.
Gifted Mathematician	A person with mathematical giftedness.
Mathematical Giftedness (2.4.2)	A quality of giftedness in mathematics.
More Knowledgeable Other (2.6)	A person perceived by a learner as highly skilled in the present task, and against whom they can therefore judge their Zone of Proximal Development.
Specialist Mathematics School	A 16-19 free school in which all students study A-Level Mathematics and Further Mathematics, in addition to another mathematical science.
Talent (2.4.1)	Knowledge and skills held by a person, with no assumptions made regarding how the knowledge or skills were developed by that person.
Typically-Developing Learner	A learner who is not a gifted learner.
Typically-Developing Mathematician	A person who is not a gifted mathematician, although they might be a gifted learner in a different subject or domain.
Quality of Giftedness (2.4.1)	The ability of some individuals to make especially fast progress in a specific subject or domain.
Year 12/Year 13	Year 12 is the first year of education subsequent to the completion of GCSEs. Learners usually turn 17 in Year 12. Year 13 is then the following academic year.

Zone of Proximal Development of a Gifted Mathematician (2.6)	The cognitive position between what a gifted mathematician currently perceives as unfeasible for them even with support from another person who is highly skilled in the task, and that which they are already capable of achieving (largely or entirely) unaided.

Abbreviations and Acronyms

Abbreviation/Acronym	Meaning
BPhO	British Physics Olympiad
FE	Further Education (in relation to the FE sector, and not the further education phase)
MAT	Mathematics Admissions Test
MKO(s)	More Knowledgeable Other(s)
NSAA	Natural Sciences Admissions Assessment
STEP	Sixth Term Examination Paper
ZPD(s)	Zone(s) of Proximal Development

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1 Introduction: Research with Gifted Mathematicians During the Further Education Phase

1.1 Introduction

The purpose of this chapter is to outline the issues faced by gifted mathematicians during their further education phase as the motivator for the research represented by this thesis. To do so, it briefly introduces my personal and professional background with gifted mathematicians, and the literature that pertains to the nature of their challenges, to contextualise the research. The salient aspects of the study into gifted mathematicians' experiences throughout their further education phase are presented, and an explanation of how the thesis is structured to justify the ultimate creation of a pedagogical model for supporting advanced mathematical-development given.

1.2 The Status of Gifted Mathematicians During Further Education

The policy landscape which has increasingly prioritised the study of mathematics across the gamut of institutions both within the vocational Further Education (FE) sector and which deliver programmes throughout the further education phase. Against policy intentions that are seeking to make the study of mathematics compulsory for all to age eighteen by as early as 2025 (Lewis & Maisuria, 2023), today's gifted mathematicians have been progressively deprioritised. Mathematics departments in FE institutions must channel their resources into providing for those resitting GCSE Mathematics to satisfy post-sixteen funding regulations (ESFA, 2024). Moreover, gifted mathematicians' pursuits and achievements beyond A-Level grades are not accounted for when assessing an FE institution's quality (DfE, 2024), with outcomes assessed based on 'value added' beyond projected final grades. With finite resources and little policy incentive to provide specialised support to gifted mathematicians, such learners regularly find themselves without sufficient dedicated support when pursuing advanced mathematical-development (Glossary). Throughout this pursuit, many gifted mathematicians will experience the feeling of conceptual difficulty when developing as mathematicians for the first time (Siklos, 2019). A common lack of appropriate mathematical-challenge prior to their further education often means gifted mathematicians have had few opportunities to develop resilience when feeling challenged (Avhustiuk, Pasichnyk & Kalamazh, 2018). This creates a perfect storm, often resulting in many forgoing their higher ambitions in their further education phase altogether. This research therefore seeks to better understand the challenges associated with advanced mathematical-development and to improve education for gifted mathematicians throughout their further education. To do so, it has a dual nature. It is explorative in that it places a lens on the perceptions of gifted mathematicians, that they might feel valued in a sector that they often perceive ignores their needs. These perceptions are subsequently interpreted to inform the creation of a pedagogical model (6.3), for use by mathematics who support advanced mathematical-development throughout the further education phase.

1.3 The Research Questions

The research questions driving this study are stated as follows:

1. How do gifted mathematicians perceive their experiences of advanced mathematical-development throughout the further education phase?
2. What implications do gifted mathematicians’ perceptions of advanced mathematical-development have for effective pedagogical approaches which support them through the challenges they associate with this experience?

1.4 The Core Literature, Activities, and Procedures

There is bountiful literature pertaining to mathematics during further education (Dalby & Noyes, 2016, 2018, 2020; Nixon & Cooper, 2020; Cogan, Schmidt & Guo, 2019), giftedness during further education (Smith & Wood, 2020; Naif, 2019; Muratori & Smith, 2018), and mathematical giftedness (Smothers *et al.*, 2021; Leikin, 2020; Daikou & Telfer, 2018; Singer *et al.*, 2016). However, very little exists which specifically relates to mathematical giftedness during further education and the strategies through which it can be nurtured. Siklos' (2019) work based on his abundant experiences of supporting many applicants to elite universities to develop mathematical skills at admissions-assessment level between 1987 and 2019 is the notable exception, and his perspectives are subsequently applied extensively throughout this thesis. However, he nurtured advanced mathematical-development predominantly through his role in Higher Education outreach and as an examiner (CTC, 2024). While valuable, his perspectives are not therefore the result of developing mathematical giftedness from a teaching role inside the FE sector. Moreover, they are borne of his own professional development based on extensive reflections on working with gifted mathematicians during their further education. They are not, however, the culmination of empirical research which sought gifted mathematicians' perspectives of developing as mathematicians throughout their further education while they were living through that experience.

The knowledge gaps outlined above were addressed in the research by inviting three gifted mathematicians from my institution (see 1.5) to take part in a series of problem-solving sessions. Alongside this, they recorded their perceptions of developing as gifted mathematicians via digital diaries and semistructured interviews. The sessions began in January 2023 and took place throughout the academic year. Problem-solving was established as a common development area for gifted mathematicians throughout their further education (Chytrý *et al.*, 2020; Ngiamsunthorn, 2020; Kozlowski & Chamberlin, 2019; Kozlowski & Si, 2019) through which their perceptions of appropriate challenge within advanced mathematical-development could be formed. Problems beyond typical A-Level challenge subsequently became activities which took place during the sessions. Five scaffolding techniques were developed through further developing existing perspectives pertaining to the support of problem-solving (3.9.2; GMI, 2019; Khong, Saito & Gillies, 2019; NRICH, 2021; Szabo *et al.*, 2020; Wrightsmant, Swartz & Warshauer, 2023), and subsequently trialled within the sessions. Participants' perceptions of these techniques, and wider influences within advanced mathematical-development, were recorded at three points. They kept digital diaries (3.10.1) for a fourteen-day period in January 2023, making their first entries during the first session. Moreover, they made entries throughout a second fourteen-day period in June 2023, and took part in interviews (3.10.2) in December 2023 and January 2024 to further reflect on their experiences. Interpretative phenomenological analysis (Delve & Limpaecher, 2023) was refined to facilitate an analytical procedure (3.12) which established how each individual's perceptions of advanced mathematical-development evolved over time (4.2, 4.3, 4.4). This procedure also ensured their individuality was valued (Squires, 2023) when their respective perceptions were analysed collectively to create the shared themes within the critical analysis (5). Vygotsky's (1978) theory of the Zone of Proximal Development (ZPD) was refined (2.6) for application as a framework to facilitate this analysis, which evaluated the social mechanisms through which each participant developed as a mathematician. This led to an improved understanding of advanced mathematical-development and how to support it, enabling the established scaffolding strategies to be situated within a wider pedagogical model (6.3) which reflects the evolution of gifted mathematicians' support needs.

1.5 Personal and Professional Context: Gifted Mathematician and Teacher

My desire to work closely with gifted mathematicians during their further education phase is primarily motivated by my personal experiences as a gifted mathematics student and professional experiences as a gifted mathematician teaching other gifted mathematicians in this phase. These experiences position me uniquely to undertake doctoral research into strategies to better support gifted mathematicians to maximise their potential. A brief overview is therefore inherently necessary to faithfully represent and contextualise my research. A full timeline of biographical events can be found in Appendix One.

My mathematical giftedness was identified at age eleven, shortly after starting secondary school. I had always found maths straightforward at primary school.

However, I was attaining highly in all subjects which masked my particular predisposition for maths. It therefore took a mathematics specialist to perceive my potential. I was extremely fortunate that one attentive teacher did so and began to mentor me. However, my recollection is that most teachers thought acceleration was a bad idea, and so actively tried to impede it. This is perhaps one example of a recognised phenomenon in gifted education, that teachers inexperienced in this practice area hold unhelpful beliefs which motivate them to act in ways which are not only unsupportive towards their gifted students (Matheis *et al.*, 2017), but potentially actively harmful (Fetterman, 1988). In particular, many hold the false belief that acceleration is psychologically harmful, which has been shown to be false (Bernstein, Lubinski & Benbow, 2020). My mentor, however, understood the type of support I would benefit from, and helped me advocate for the resources I needed to pursue my academic ambitions. So, despite ongoing resistance from some teachers, I achieved A* in GCSE Mathematics at age thirteen.

I attended an 11-16 school which did not offer A-Level courses. However, I intended to commence A-Level study in the subsequent academic year, and was therefore permitted to visit my local sixth form college for two teaching hours per week. I was initially prevented from studying A-Level Further Mathematics (OCR, 2024a). Thankfully, teachers at the college supported me with resources to self-study additional content. I was entered for two AS-Level Mathematics modules in January 2008, achieving 100% in both. The college's principal then took a personal interest, providing additional support to my plan to take a full complement of three A-Levels and transition fully into FE. I did so the following September. However, at this point I had only studied two AS-Levels, so, despite having achieved two A grades in these qualifications, my opportunities to apply for undergraduate courses were limited. I overcame this by self-teaching both AS- and A-Level Statistics during Year 13. My application to read mathematics at Magdalene College, Cambridge was ultimately successful, and I received a conditional offer at age fourteen. I particularly enjoyed independent learning when at first the material felt challenging, feeling excited to puzzle through difficult problems. I relished the times I did not immediately master them, learning the most when I felt stuck. I enjoyed this immensely and often sought out difficult problems independently. This was instrumental in my exceeding the conditions of my offer when I took the entrance exams in summer 2009. I took up my place in October at age fifteen, and went on to complete master's study by age nineteen. Despite feeling immense pride for everything I achieved in Year 13, I also acknowledge much frustration. I felt my elevated workload could have been avoided had I been allowed to study for three A-Levels from the beginning of Year 12. I therefore felt the impact of ineffective teaching practices in the long-term way described by Papadopoulos (2020) as a common outcome in the education of the gifted. Fortunately, my mentor, and, subsequently, supportive teachers at college, gave me ample guidance. With their help I was ultimately able to make up the lost ground and pursue my desired academic endeavours within my aspirational timelines.

Despite the ultimate success of the endeavours I pursued throughout Year 12 and 13, I developed a sense that gifted mathematicians might not get the support they require to achieve their full academic potential throughout their further education. Moreover, I recognised that my earlier feelings of contentment during my further education phase might not be shared by other gifted mathematicians during this phase. In particular, I realised my feelings of excitement when appropriately challenged could easily have been felt negatively were I not proving up to the challenge independently. I had mentors who invested significant time in me as an individual. However, this was not part of the standard provision; it was put in place specifically to support me. Benefitting from this support was a combination of being fortunate enough to connect with the right teachers at the right times, and my dogged determination to realise my full potential. Gifted mathematicians without resolute determination to succeed might never even ask for better support. I therefore reasoned there could be many neglected gifted mathematicians in their further education phase who might be receptive to my effort to re-engage them in advanced mathematical-development. Moreover, I recalled my further education as the phase I naturally knew how to thrive in as a gifted mathematician, using teacher support to facilitate my independent exploration of appropriately-challenging activities. So, thinking my experiences could be of value to other gifted mathematicians in Years 12 and 13, I decided to train on an post-compulsory education and training pathway (Machin *et al.*, 2023, 2024). I secured a teaching placement at the sixth form where I studied and was delighted to be trusted with incredible latitude to design activities for gifted mathematicians so early in my teaching career. This opportunity meant I was able to develop a specialised pedagogical skillset (Hanley & Thompson, 2021) of benefit to gifted mathematicians. It also led to the undertaking of a PGCE research project which would go on to inspire this doctoral research.

Since the study's inception, I have moved into my current role at a specialist mathematics school. Mathematics schools are 16-19 state-funded free-schools specialising in developing mathematical talents (DfE, 2022; Borovik, 2012). Although referred to as a school, this should not be interpreted as a reference to the educational phase. Mathematics schools are most similar to standalone sixth form colleges in that their students are usually undertaking academic (A-Level) rather than vocational programmes. However, mathematics schools are much smaller, typically educating between just 100-140 students across Years 12 and 13. All students in mathematics schools take A-Level Mathematics and Further Mathematics (OCR, 2024a, 2024b). This study is therefore an investigation of my own practice with several gifted mathematicians in my current institution, a specialist maths school, a niche institution within the FE sector.

1.6 Thesis Structure

This thesis is presented as a six-chapter document. The introduction serves to outline the issues faced by gifted mathematicians during their further education. It also explicitly presents my experiences as a gifted mathematician, to provide sufficient context for the reflexive approach adopted in later chapters towards the creation of a pedagogical model (6.3) for supporting advanced mathematical-development.

The literature review chapter begins by critically evaluating views from the wider fields of Giftedness, Mathematics, and Further Education to analyse the challenges in applying the associated literature within this doctoral research. In doing so, a substantial literature gap at the triadic intersection of these fields is established. Mathematical giftedness is situated within the context of the further education phase, enabling perspectives from the literature pertaining to injustice towards gifted learners to be analysed in relation to this specific subject and educational phase. Potential reasons for difficulties gifted mathematicians may face throughout their further education are subsequently evaluated, leading to recommendations for apposite avenues of investigation, in particular of gifted mathematicians' experiences in FE and of effectual pedagogical practice to support them. The chapter concludes by identifying a theory upon which the nuances of advanced mathematical-development can be critically evaluated. Vygotsky's (1978) theory of the ZPD is subsequently introduced as an apposite theoretical framework, and further refined for application to gifted mathematicians in a variety of FE institutions. In particular, the ZPD is redefined for this purpose, and the vocabulary for describing a gifted mathematicians ZPD throughout their further education is developed.

The methodology chapter begins with a brief narrative of the activities undertaken throughout the study (3.6). This serves to set the scene for the subsequent sections, which give the methodological justification and provide the detail around how the study was designed, in particular the nuances of how digital diaries and the novel data-analysis procedure were developed and subsequently utilised.

The findings, analysis, and discussion are presented in two chapters. In the Findings chapter, the findings are presented through detailing the experiences of each participant on an individual basis, beginning with a brief biography of each participant to aid the contextualisation of their subsequent views. Taking the participants in turn, the Findings chapter therefore charts how each perceived their advanced mathematical-development as they progressed from novices to mature problem-solvers. The individual's journey of advanced mathematical-development within the study is then summarised to conclude their section. Discussion and Analysis chapter follows, detailing the shared themes pertinent to all participants, and analysing the nuances within them to create a pedagogical model (6.3) of support during advanced mathematical-development.

The Conclusions and Recommendations chapter begins by summarising the contributions to knowledge made by the study, explaining how these contributions answered the research questions. The pedagogical model (6.3) of support through advanced mathematical-development is also presented in detail. The thesis then concludes by considering the model's limitations, making recommendations for future research, and reflecting on the professional learning which resulted from this doctoral work.

1.7 Summary

This chapter has summarised the research undertaken to further investigate potential solutions to the issues faced by gifted mathematicians during their further education, and briefly outlined my personal and professional background. The Literature Review chapter that follows will critically evaluate the scholarly knowledge which exists within this niche as the starting point for future research.

2 Literature Review: Mathematical Giftedness During Further Education

2.1 Introduction

The original motivators of the research discussed in the Introduction chapter (1.5) were professional observations that gifted mathematicians often find the feeling of appropriate challenge during their further education so uncomfortable they decide not to pursue it. This Literature Review chapter will make this more rigorous by synthesising the perspectives on gifted mathematicians' experiences of developing during their further education which are prominent in the literature. The review begins by narrowing the perspectives on concepts of giftedness into notions of mathematical giftedness, then situates mathematical giftedness in the context of the further education phase. In doing so, the nature of the challenges gifted mathematicians are likely to face during this phase is established to critically evaluate the type of support they might need. A substantial literature gap is then identified, which pertains to the current absence of a pedagogical model and associated scaffolding strategies for supporting gifted mathematicians' development during their further education. The chapter concludes by establishing the implications of utilising the quality of giftedness theory to conceptualise gifted mathematicians for the theoretical framework which can be applied to critically evaluate the nuances of advanced mathematical-development. Vygotsky's (1978) theory of the ZPD is subsequently identified as apposite, and honed for application to gifted mathematicians during their further education, in particular by redefining the ZPD and developing suitable vocabulary to describe the ZPD and its evolution in this specific context.

2.2 Gifted Neglect in the Literature

Discomfort when developing problem-solving skills is to be anticipated in mathematics learners in general (Halmo, Yamini & Stanton, 2024). However, gifted mathematicians should find this struggle productive during their further education, and hence, perceiving their discomfort as an indicator of potential progress, enjoy feeling challenged in this way (Siklos, 2019). Professional observations have also suggested that gifted mathematicians require more support than is routinely available from institutions delivering further education to work through this discomfort positively. The individual institutions and gifted mathematicians through which this has been observed constitute specific examples of a much wider issue. Many authors worldwide acknowledge a longstanding and unhelpful attitude formed of misconceptions about gifted people. In particular, that the gifted will attain highly without support, and that they are subsequently less entitled to educational resources than their typically-developing peers (Wyllie, 2019; Merry, 2008; Feldhusen, 1989; Witty, 1958). This attitude has been utilised to justify depriving gifted learners of specialised support; Fetterman (1988) chronicled its affects in a variety of settings. He (*ibid.*) decried it not just as neglectful, but as academic abuse perpetrated by insecure institutions and teachers who feel inadequate in the company of highly-capable young people and so, actively or subconsciously, sabotage their success. Many teachers countered that they simply did not know how to help gifted learners effectively. Sustaining their own ignorance rather than actively seeking professional development was considered by some to suppress gifted learners (*ibid.*), who held teachers nonetheless accountable. Opinions have somewhat mellowed over time. However, attainment gap discourses which celebrate *'raising the floor'* have also seemingly been misinterpreted to justify leaving the *'ceiling'* in place (Wyllie, 2019). Hence, discourse of this nature has not been described as a lack of awareness, but as a form of purposeful negligence towards the gifted (Riley, 2019). Narratives which suggest gifted people are less worthy of resources without fully acknowledging their particular challenges have persisted (Finn, 2014). Mainstream settings therefore commonly justify the withholding of gifted provision, leaving such learners under-challenged. Merry (2008) considered this to be a direct example of intentional gifted neglect. Moreover, Shilvock (2017) observed gifted students feeling unchallenged going on to forgo the pursuit of their maximum potential. This foreshadows the presence of gifted mathematicians arriving in FE institutions having felt underchallenged for a long time and who have already resigned themselves to lesser achievements (2.5.3). Gifted girls (Boston & Cimpian, 2018), and those from lower socioeconomic backgrounds (Wai & Worrell, 2020; Passow, 1972), are the most affected in this way given their access to Higher Education is already disproportionately low (Harrison & Waller, 2018). Subsequently, gifted neglect has been interpreted as a form of societal neglect (Wai & Lovett, 2021), complexifying the process of recruiting effectively to careers in the mathematical sciences (Petry, 2019) which then leaves society bereft of the value of this labour.

2.3 Giftedness, Mathematics, and Further Education

Giftedness, Mathematics, and Further Education are the three predominant overarching fields pertaining to the target group. In particular, knowledge contributed will reside in the niche at their triadic intersection. Many studies have investigated dyadic conjunctions of these fields (namely Mathematics in FE, Giftedness in FE, and Mathematical Giftedness). However, very few have broached all three simultaneously by exploring Mathematical Giftedness in FE. Locating the sparse available literature within the triadic intersection will be one key aim of this literature review. However, it must also look more-broadly. Unfortunately, inquiries pertaining to the three dyadic conjunctions tend to occupy positions which distinguish them from the third field in some significant way. Unsurprisingly given the relative abundances of such learners (2.5.1), FE Mathematics literature prioritises numeracy (BSA, 2001) and functional skills for typically-developing mathematicians (Dalby & Noyes, 2020; Nixon & Cooper, 2020), who commonly resist additional mathematics education (Bellamy, 2017), to prepare them for further study (Cogan, Schmidt & Guo, 2019) and careers (Allan, 2017). Indeed, pedagogies for students resitting GCSE Mathematics are the current focus of the most-prominent national study into FE Mathematics (MM, 2024). Likewise, literature pertaining to giftedness in FE commonly connects giftedness across a multitude of subjects, seeking to support gifted people to pursue shared goals (Muratori & Smith, 2018), in particular by providing progression and career guidance (Smith & Wood, 2020; Naif, 2019). Subject-specific perspectives fall short of offering specialised support for gifted students during their further education, instead favouring activity differentiation (Smothers *et al.*, 2021; Hall, 2018) in mixed-ability classrooms (Daikou & Telfer, 2018; Dixon & Pilkington, 2017). Finally, mathematical giftedness literature commonly seeks to explore the nature of giftedness (Szabo, 2018; Leikin *et al.*, 2017) from individual perspectives (Leikin, 2020; Leikin, Leikin & Waisman, 2018; Singer, Sheffield & Leikin, 2017), not to investigate the development of groups of such learners (Barraza-García, Romo-Vázquez & Roa-Fuentes, 2020) within a specific educational phase. Studies researching gifted identification also generally prioritise early identification (de Vreeze-Westgeest & Vogelaar, 2022; Zubova *et al.*, 2021; Dunn, Georgiou & Das, 2020; Al-Hroub & Whitebread, 2019). However, pedagogical practices must acknowledge that 16-19 learners are likely to have different needs than children (Machin *et al.*, 2024, 2023) and, that gifted mathematicians might be particularly disengaged at the onset of the further education phase (2.5.3).

2.4 Understanding Mathematical Giftedness

2.4.1 Concepts of Giftedness

Conceptualisations of gifts and talents fundamentally influence how giftedness is identified and developed (Matthews, Subotnik & Horowitz, 2009). Both the educational setting and academic subject underline similarities between subsections of gifted people (Singer *et al.*, 2016) to assess shared challenges and notions of pedagogical effectiveness (van Tassel-Baska, 2018, 2021). Determining an appropriate conceptualisation is therefore an imperative initial consideration. Gifted programmes have been continually criticised by many scholars and practitioners as a form of elitism (Radulović, 2022; Matthews, 2014; Howley, 1986), designed to unfairly advantage elite learners with an already-high aptitude for learning. This is vehemently contested by the body of giftedness research (Alodat, Ghazal & Al-Hamouri, 2020; Dai, 2018), which has exposed widespread underachievement in gifted people (Tan, Tan & Surendran, 2021; Alexopoulou, Batsou & Drigas, 2019), with seminal studies (Freeman, 1998, 2012) highlighting that gifted people need nurturing differently than their typically-developing peers. Care must nonetheless be taken to evaluate the potential stigma such a label might attach to gifted people (Worrell, 2009) when determining this conceptualisation, either intrinsically because of, or socially by, those who perceive the labelling as the creation and prioritisation of an elite class (Dodillet, 2019).

The phrase *'gifted and talented'* has been prominent in UK discourse (Koshy, Portman Smith & Casey, 2018). However, much debate surrounds the nature and effective nurture of gifts and talents. Dictionary definitions only partially capture the essence of this discussion. Where both *'gift'* and 'talent' are defined as *'a natural ability to do something well'* (OLD, 2024a, 2024b), a gift is also defined as *'a thing given willingly, without payment'* (Lexico, 2024a). Talent ascribes no notion to how knowledge and skills were obtained, whereas gifted suggests they were obtained with no effort. The theoretical perspectives on giftedness and talent therefore occupy positions on a continuum between *'intrinsic gifts'* and *'developable talents'*, the two predominant opposing schools of thought (Branton Shearer, 2020; Worrell & Erwin, 2011). Many argue talents are inherently developable, and that, when sufficiently practised (Ericsson, Krampe & Tesch-Römer, 1993) with focused support (Biech, 2018), anybody can develop skills at the highest level in any subject area (Ericsson & Pool, 2016). Proponents of this viewpoint further argue that gifted identification is a wasteful practice (Dhaliwal & Hauer, 2021) and that efforts should be channelled into developing talents rather than searching for gifts (Berzsenyi, 2019). The primary goal of social justice, that people's socioeconomic background and upbringing become irrelevant to their potential (Smith, 2018), and hence that anybody's success can be elevated, are intrinsic to this conceptualisation which therefore enjoys much societal support (Rasmussen & Lingard, 2018). However, Dai (2020) suggested that individual differences in potential exist across all society. Moreover, he propounded that disregarding these differences deprives such learners from lower socioeconomic backgrounds disproportionately, advocating for widening access to tailored gifted programmes. Supporters of intrinsic gifts, the juxtaposed viewpoint, argue gifts are unique to the individual, lifelong, and inherent from birth (Freeman, 1998, 2012; Sternberg, 2018a). Such a position asserts that opposing notions of talent creation disrespect individuals by seeking to manoeuvre them into societally-determined roles rather than valuing who they intrinsically are and nurturing them on that basis (Peters, Carter & Plucker, 2020). Attempts to deal with both concepts collectively are therefore criticised for homogenising the ideas (Dai, 2018). Consequently, terms like *'gifted and talented'* have been decried as too vague to accurately represent either group, rendering associated provisions suboptimal for either (Horn, 2019).

The harmonious coexistence of developable talents and intrinsic giftedness relies on the critical distinction between *'inborn gifts'* and a natural *quality* of giftedness. Accepting that anybody with sufficient determination and focus can achieve mastery of high-level knowledge or skills (Sella & Cohen Kadosh, 2018) does not preclude the existence of people who find this significantly easier than their typical peers. The ability to process information in a more expedient and complex way (Feldhusen, 1989) than one's peers has been observed both within cognitive styles (Solé-Casals *et al.*, 2019) and via differences in neuroanatomy (Kuhn *et al.*, 2021). A proclivity towards studying areas of interests also facilitates ongoing engagement, aiding the embedding of associated knowledge and skills in the memory (Banikowski & Mehring, 1999). As people with this ability must still develop knowledge and skills, it has not been characterised as a gift, rather a *'quality of giftedness'* (Freeman, 2012) which enables rapid progress in a particular domain (Sternberg, 2018a; Freeman, 2013). The *'gift'* in giftedness therefore refers to the ability to make atypically-fast progress, not the knowledge and skills in their own right, antithetically to the notions of talent (2.4.1).

2.4.2 A Quality of Mathematical Giftedness

Gifted mathematicians have been observed demonstrating sustained commitment to mathematical development alongside natural flair and interest (Leikin, 2020). Mathematics uses abstract logic to establish universal laws which apply in many contexts (Whitehead, 2017). Historically, eminent mathematicians have conceptualised phenomena in seemingly inconceivable ways (Stillwell, 2020), differentiating themselves from peers. Their singular ways of thinking have undoubtedly accelerated the subject's progress (Wilson & Flood, 2020; Wittmann, 2020). For example, many consider Ramanujan's[1] contributions to have outstripped those of his contemporaries despite his relative lack of formal training (Rao, 2021). Famed mathematicians' surnames are punctuated throughout mathematics' history (Gifford & Young, 2021), which comprises theorems, proofs, and ideas titled eponymically by the mathematical community to honour the elite mathematicians responsible for contributing them (Başibüyük & Şahin, 2019; Heaton, 2017).

It has been suggested mathematics should be both considered as (Dyck, 2020; Ranta, 2020), and learned like (Vukovic & Lesaux, 2013), a language. However, Dabell (2022), in his agreement with this view, held that *all* subjects form unique languages, but that mathematics is particularly distinctive given its use of symbols to codify abstract ideas. It might therefore be argued mathematical progress depends on sustained regular practice in the same way learning a language might (Babayiğit & Shapiro, 2020). However, this characterisation is contradicted by the existence of many individuals contributing new knowledge to the subject without this effort. Peterson (2020) noted that typically-developing children conversing without proper vocabulary or syntax can often still be accurately comprehended, and hence that rudimentary language serves the same purposes. However, such children might not be able to communicate abstract concepts or be understood outside of a specific context. Nevertheless, the early stages of language learning enjoy some defence to ambiguity in ways mathematical learning does not. Mathematical clarity depends on rigour and precision (Richards, 1991) to remain unambiguous (Baldwin, 2016). Gifted mathematicians exhibiting this clarity without formal training are hence as remarkable

[1] Ramanujan is a renowned 20th century Indian mathematician. Having never been formally trained, he was able to derive thousands of results and theorems so novel that many leading mathematical scholars were unable to perceive his genius in them. They lacked either the effort or the capacity to understand Ramanujan's work. He made significant contributions to mathematical analysis and number theory which were largely discovered during his independent work in India prior to his eventual work at the University of Cambridge.

as young children using precise grammar and vocabulary to concisely convey complex ideas. Thus, gifted mathematicians are perhaps predisposed to marshalling their thoughts in meticulously-structured ways, orienting their minds with the subject itself (Chassy & Grodd, 2016), and hence could be perceived as members of the mathematical elite. Likhanov *et al.* (2020) supported this view, finding that high-achieving adolescents in the mathematical sciences are likely to possess similar combinations of behavioural and personality characteristics. Moreover, Parish (2014) considered the nature of these characteristics instinctual, echoing wider notions that successful mathematics students have similar cognitive styles (Susandi *et al.*, 2019). This was further described by Riley (2021) as the 'like-mindedness' of gifted people, which Singer, Sheffield, and Leikin (2017) believed naturally distinguishes them from their typically-developing peers. The idea of a quality of giftedness therefore initially appears to better conceptualise the gifted mathematicians central to this study.

Many gifted mathematicians have been noted dedicating prodigious effort to their advanced mathematical-development in the literature (Leikin *et al.*, 2017). This appears to align better with the talent-development perspective, raising one concern around rejecting it prematurely. However, some theorists do not consider the two perspectives to be mutually exclusive, rather just opposing points on a continuum that can, to varying extents, coexist (Worrell *et al.*, 2019). Talent-development approaches have found their roots in the gifted and talented education literature. However, such approaches have been suggested as broadly-applicable to a variety of learners (Dai, 2019). Gagné (2018) described talent development as concerned with achievements in their own right without any particular reference to natural affinity, indicating that a person possessing a natural quality of giftedness is not precluded from actively developing their knowledge and skills in analogous ways. Subotnik, Olszewski-Kubilius, and Worrell (2018) went further, suggesting gifted achievement is *reliant* on an underlying gift being actively developed. Moreover, it has been argued this development transcends natural affinity as gifted individuals require organised external support to remain motivated (Burns & Martin, 2021) in either case. Those possessing a quality of giftedness therefore can, and do, benefit from purposeful practice, and support to sustain such practice, in their efforts to maximise their potential (Sternberg, 2018b).

How gifted mathematicians feel towards investing effort also affects their success (Sella & Cohen Kadosh, 2018). This is influenced by the attitudes of various people including parents (Ruf, 2020), teachers, and mentors (Paik, Gozali & Marshall-Harper, 2019). Mazana, Montero, and Casmir (2018) further contended that educational environments and teachers' instructional practices shape learners' attitudes towards developing mathematically. Such attitudes throughout secondary education were therefore considered by Kay (2020) to be many and varied, influencing the foundation upon which learners transitioning into the further education phase build upon. This suggests that gifted mathematicians might begin their further education with a plethora of levels of confidence, abilities, and outlooks. It also highlights the importance of distinguishing between developed mathematical-talents and the quality of mathematical giftedness.

Typical mathematics practitioners supporting the further education phase might find it challenging to correctly distinguish between learners with developed mathematical-talents and those with a quality of mathematical giftedness (Maker, 2020). In particular, perceiving them simply as the most elite in their institution on the sole basis of high mathematics attainment (Budínová, 2024) obscures the finer distinctions between the two groups. For example, those with a quality of mathematical giftedness might not always be the hardest working (Sella & Cohen Kadosh, 2018), nor even keen to participate in mathematics activities (Zavala Berbena & de la Torre García, 2021), unless the content is appropriately challenging for them (2.5). Practitioners might therefore employ ineffective identification strategies which wrongly attribute a quality of giftedness to those who have simply invested significant time and effort to develop their mathematical talents (Ericsson & Pool, 2016). In particular, high-attaining mathematicians might appear to hold a quality of giftedness in that they are judged by a practitioner to be elite in comparison with lower-attaining peers, who make up the majority of learners in the classroom. However, this is an invalid assessment of a quality of giftedness, usually judged based on a learner's ability to develop knowledge and skills during activities associated with A-Level curricula, which should be underchallenging to the mathematically gifted. To validate this judgement, it should be made in the context of mathematics activities from beyond the A-Level curricula. Those high attainers who are both keen to participate (Deringöl, 2018) and continue to develop expediently when engaged in such activities are more likely to have a quality of mathematical giftedness. Hence, this is one means through which a gifted mathematician might be identified in practice by a typical practitioner in practice.

2.5 The Plight of the Mathematically Gifted During Their Further Education

2.5.1 UK National Policy Influences on Mathematical Giftedness During Further Education

Gifted neglect is not consistently countered by the introduction of gifted policies in schools. Where national policies are absent, many school leaders have resisted their teachers' calls for additional specialised support (Cross, Cross & O'Reilly, 2018). However, Koshy and Robinson (2006) found gifted neglect persisting even during times gifted policy was prominent in national educational landscapes. The UK does not currently benefit from legislation nor national policy in this regard (Loft, Long & Danechi, 2020; Koshy, Portman Smith & Casey, 2018). Giftedness was first recognised within UK national policy when the *Education Act 1944* introduced the tripartite system, separating students out into secondary modern, secondary technical, and grammar schools depending on their performance in the transfer test (11+). The potential of all students in professional and academic qualifications was assessed. Those with high academic potential were subsequently educated in grammar schools, homogeneous high-ability learning environments. This was, at the time, considered the epitome of social justice given that it created previously non-existent pathways into academised professions for children from lower socioeconomic backgrounds (BE, 1941). The threefold provision was maintained as recently as the

1960s (*Education Act 1962*). However, during this period the 11+ was found to have poor psychological foundation and to be essentially sorting students by their socioeconomic backgrounds (Koshy & Casey, 1998), undermining its use. Local authorities were guided to reorganise secondary education, giving rise to the state comprehensive school (DES, 1965) which subsumed gifted learners into mixed-ability educational contexts. The national prominence of gifted education has ebbed and flowed since that time (Sutherland & Reid, 2023). Despite the final covenants of the *Education Act 1944* being repealed in its *1996* update, New Labour assumed power soon thereafter. They desired to facilitate better access to Higher Education and training for those from lower socioeconomic backgrounds. Their *Excellence in Cities* (DfEE, 1999) initiative is the most recent national policy providing for gifted learners in comprehensive schools, and its repeal in 2010 marked the conclusion of the only period in UK history when *all* schools were required to cater specifically to gifted learners. However, its implementation has been described as an elaborate sleight of hand, where policy discourse emphasised gifted learners to mislead society into thinking the gifted were being prioritised. Associated national structures (GTU, 2010), and aims to ensure every school benefitted from a Gifted and Talented Coordinator, promised much (Maddern, 2009). However, the policy ultimately prioritised lower socioeconomic areas to raise attainment society-wide, not ensure gifted people had tailored provision specifically (Smithers & Robinson, 2012; Tomlinson, 2005). Not only were gifted learners deprioritised within policy supposedly meant to champion them, but such policy was utilised to provide greater support to typically-developing students. This revalidated the attitude that gifted people are less deserving of resources nationally, further ingraining the neglect and sustaining institutions' ignorance to widespread gifted underachievement (Tan, Tan & Surendran, 2021; Gottlieb, 2020).

The impact of the longstanding absence of national gifted policy on gifted mathematicians is further hindered by FE sector policy. Funding for courses in FE institutions is conditional on them having passed GCSE Mathematics. If they have not, they must re-take it alongside other programmes for the institution to be awarded the funding for the entire course of study (ESFA, 2024). Likewise, a gifted learner's attainment beyond the A* threshold does nothing to improve an institution's performance measure (DfE, 2024). It is therefore unsurprising when FE mathematics departments concentrate their resources into typically-developing learners, especially as the scarcity of gifted mathematicians renders them easily overlooked. An average-size cohort hosts just two mathematicians pursuing higher ambitions like elite university admissions tests (AoC, 2024; Jadhav, 2010; UoC, 2024a; UoO, 2024). These approximately 500 gifted mathematicians across all FE settings in England each year make up just 0.58% of A-Level Mathematics students. The minority is therefore obscured by the 36000 typically-developing A-Level Mathematics students alongside them who achieve A*/A grades, and the 20000 who achieve D-U grades (GDS, 2024). Students resitting GCSE Mathematics are also significantly more bountiful than gifted mathematicians, making up another 154000 (JCQ, 2023). Moreover, current A-Level Mathematics and Further Mathematics specifications do not specifically prioritise developing creative problem-solving skills (OCR, 2024a, 2024b). Instead, they focus on topic knowledge and the application of familiar methods within them. They do not, therefore, challenge mathematicians to formulate their own solutions to problems. A-Level syllabi have faced much public opprobrium on this basis (Bentley, 2019; Turner & Somerville, 2019; Buckland, 2017; Sellgren & Richardson, 2017; Ward, 2017) since A-Levels were reformed into linear qualifications in 2015. This is despite such reforms intending to develop skills problem-solving skills in their guise of applying mathematics to wider contexts (Ofqual, 2018).

2.5.2 A Quality of Giftedness as Educational Disadvantage

A quality of giftedness intellectualisation must overcome the common view that inborn giftedness is a form of privilege. Parekh, Brown, and Robson (2018) argued that no correlation existed between those identified as gifted and those attaining highly, and that gifted identification practices favoured those from traditionally-privileged backgrounds. This might be interpreted as a criticism of several factors of gifted identification. Predominantly, their (*ibid.*) original argument was that those from privileged backgrounds are more likely to have enjoyed a wide variety of developmental experiences which a particular quality of giftedness could resonate with, and hence make itself more-obviously apparent. Such students have access to additional resources which enable faster learning, creating the appearance of expedient progress relative to peers in the classroom (Hodges *et al.*, 2018). They are also more likely to be put forward for consideration by parents (Mollenkopf *et al.*, 2021), and hence potentially interact more-readily with teachers with both an interest in gifted education and the resources to prioritise it. Novak (2022) agreed that existing privilege played a role, highlighting the particular over-representation of white males. Gifted programmes therefore attract criticism for privileging people through developing attributes that may offer unfair advantages both in education and wider life (Lee, Yeo & Han, 2022). However, Robbins (2019) argued that privilege criticisms limit the wider discussion solely to the matter of gifted identification, frustrating the process of fairly and fulsomely evaluating gifted programmes' activities and outcomes. Such criticisms wrongly assume gifted programmes are universally available, and that those which already exist are effective at helping gifted learners make good progress (NAGC, 2024; Karantzas, 2017, 2019). However Peters (2022) stipulated that such programmes are commonly non-existent or ineffective.

Post (2021) described gifted learners being *dis*advantaged by those who perceive them as privileged. As gifted students often naturally realise, they are different than peers. Perceptions of giftedness as privilege would abandon them to grapple to understand their differences alone. Consequently, they often incorrectly conclude they are somehow inherently 'wrong'. This is particularly worrying given that Casino-García, García-Pérez, and Llinares-Insa (2019) found that gifted people are especially susceptible to negative mood hindering their wellbeing. The assumptions that gifted people all navigate this positively and go on to live easy and successful lives are misconceptions that have been comprehensively excoriated. The findings of Ruf (2020) suggested that a gifted person's longer-term outcomes can be hindered by parental attitudes which resist embracing their uniqueness. Furthermore, Szymanski and Wren (2019) chronicled gifted adults feeling socially isolated by their difficulty understanding and processing their otherness. This social isolation has been observed complexifying the process of gifted people adjusting to university life, causing many to academically underachieve in Higher Education (Almukhambetova & Hernández-Torrano, 2020). The associated feelings of loneliness have been described as the result of a failure to acknowledge gifted people's distinct social and emotional needs (Rinn, 2018; Rinn & Majority, 2018). Moreover, loneliness has been identified as a major risk-factor in gifted people experiencing psychological distress (Ogurlu, Yalin & Yavuz Birben, 2018), including suicidal ideation (Cross & Cross, 2021) and other symptoms of mental ill-health (Suldo, Hearon & Shaunessy-Dedrick, 2018). Moreover, it has been suggested that a quality of giftedness embeds an additional layer of psychological complexity which often isolates gifted people (Neihart & Yeo, 2018). However, this complexity can also be utilised as a tool to facilitate better understanding of themselves and others if appropriately acknowledged and nurtured (*ibid.*). The literature therefore supports the notion that, despite the vehement initial insistence to the contrary discussed in 2.2, people possessing a quality of giftedness are unlikely to thrive without aid. Hence, educational interventions are necessary to equalise their disadvantage. To do so, such strategies must support not only gifted learners' academic development, but also their social and emotional development.

2.5.3 The Breakdown of Gifted Mathematicians' Mathematical Self-Efficacy During Further Education

Section 2.5.2 echoed the perspective of Siegle and McCoach (2018) who propounded that typical provision fails to offer appropriate support to gifted learners who, subsequently, struggle to remain motivated. It is contended that a gifted learner's motivation modulates throughout typical education (Snyder & Wormington, 2020), and that underachieving gifted people struggle to sustain motivation more than their typically-developing peers (Agaliotis & Kalyva, 2019). It is therefore important to rectify an apparent contradiction: qualities of giftedness enable rapid progress, yet people possessing such mathematical giftedness underachieve during their further education. What follows sets out to evaluate how their educational experiences prior to the further education phase potentially hinder their progress, by considering the nature of appropriate-challenge within advanced mathematical-development.

Some typically-developing students will have experienced secondary education favourably, finding the level of challenge to be appropriate to their needs (Lynch, 2019) and the support offered helpful for sustaining their motivation to continue investing effort in their own development when they encounter mathematical problems of sufficient challenge during their further education. Hence, some are likely to have developed their mathematical problem-solving abilities effectively, beginning their further education not only with good grades, but ready to continue honing those skills due to their earlier education's alignment with their individual needs. However, Deng's (2019) discussion around internally high-performing education systems suggested that where highly-challenging activities are the norm in an education's culture, typically-developing students attempting to hone their mathematical talents are more likely to be discomforted by overchallenge than underchallenge. Typically-developing learners do not, by definition (Glossary, 2.4.1), benefit from any natural affinity. Hence, they need to work harder and for longer (Chinn, 2020) to master higher-level GCSE Mathematics topics (Edexcel, 2017) when pursuing top grades (Ofqual, 2017). Such learners with developed mathematical skills and knowledge therefore begin their further education with similar work ethics which have already proved successful. Typical provisions in settings delivering further education are therefore more likely to meet their needs (Eysink, van Dijk & de Jong, 2020), supporting them effectually to achieve their mathematical potential (Choy, 2021). Contrastingly, Olszewski-Kubilius and Corwith (2021) advocated for providing additional guidance to gifted learners to support their exploration of *'supra-curricular'* activities directly within their subject of giftedness but beyond the difficulty routinely encountered in mainstream curricula (UoC, 2022b). Basister and Kawai's (2018) investigation into educational practices for gifted mathematicians concurred, opining that such learners require activities beyond the classroom, which are not routinely offered, to maximise their potential. As the availability of these provisions is heavily dependent on both institutional and individual teacher practices (2.5.2), it is to be anticipated that gifted mathematicians arrive in FE settings from all manner of secondary institutions with an assortment of experiences of being appropriately challenged. Hence, their attitudes towards being challenged during their further education are likely to be equally varied. Erdogan and Yemenli (2019) observed a similar variety of attitudes in gifted mathematicians internationally. The quality of giftedness perspective therefore predicts that gifted mathematicians' attitudes will vary based on the effectiveness of their secondary education. Furthermore, it describes the gifted mathematicians being under-challenged and their subsequent underachievement. This further highlights the necessity of distinguishing between developed mathematical-talents, and qualities of mathematical giftedness. People with the latter are particularly disadvantaged by the described educational issues; those with the former are not.

Thomson (2006) advocated for the introduction of institution-wide approaches to providing gifted learners with appropriate challenge in secondary education nationwide. She (*ibid.*) suggested that the absence of appropriate challenge was a common feature of secondary education. Moreover, it has been put forward that distance learning approaches be implemented to connect gifted learners with these experiences when they are not available locally, as is routinely the case (Howley, Rhodes & Beall, 2009). Brigandi *et al.*'s (2018) investigation into enrichment programmes concurred. They (*ibid.*) found that being involved in such programmes improved gifted learners' perceptions of the quality of their education. However, they (*ibid.*) concluded that this depended on the presence of a teacher trained to provide gifted education. Mun, Ezzani, and Lee (2020) proposed that gifted learners' access to this resource has been limited by a lack of such teachers, and of teacher education programmes to create them. Moreover, it has been suggested teachers resist participating in gifted education training because they view it either as too limited to achieve its goals, or unnecessary (Kaplan Sayı, 2018). Gifted mathematicians therefore rarely experience appropriate challenge, and support to embrace it, in mainstream education in general, and throughout their mathematical learning prior to further education. Such learners routinely master typical content, which falls short of appropriate challenge, with relative ease. During further education, this trend continues throughout A-Level Mathematics and Further Mathematics which comprise many standard example-problems (Pershan, 2021). It has been reported that learners do not even need to answer many of the hardest problems to achieve A*/A grades (Bentley, 2019; Turner & Somerville, 2019; Sellgren & Richardson, 2017); grade boundaries for such grades have been set as low as 56% (OCR, 2023). The majority of students achieving such grades are therefore typically-developing mathematicians. This suggests that gifted mathematicians achieving such grades should not be considered to have realised their maximal potential during further education given that they ought, by definition (Glossary, 2.4.1), to have made faster progress.

There are many possible supra-curricular aspirations gifted mathematicians might hold during further education. Where some might prioritise success in mathematics competitions like the Olympiad (Kumar, 2023; UKMT, 2023, 2024), others might desire to conduct independent research activities (EMS, 2024; KCLMS, 2024) or pursue admission to elite universities requiring additional examination to assess mathematical progress beyond A*/A grades (UoC, 2024b; UoO, 2024). A-Level study, making up the contents of typical mathematics curricula in FE institutions, is not therefore, in isolation, an effective means of pursuing these ambitions. However, as gifted mathematicians are unlikely to have experienced much effective supra-curricular challenge previously, they have only their prior high-attainment to derive confidence from. Beek *et al.* (2017) suggested that attainment was a key mediator of positive feelings for mathematics learners. It has been put forward that mathematical self-efficacy, the concept of a student’s belief in their individual capacity to make mathematical progress (Negara *et al.*, 2021; Bandura, Freeman & Lightsey, 1999; Bandura, 1977), is improved when teachers praise high-attainment. For this reason, a link between gifted mathematicians’ motivation and attainment has been suggested (Gottfried, 2019). However, such conclusions were not made in the context of gifted programmes. Instead, these viewpoints sought to compare gifted learners with typically-developing students in mixed-ability environments, where gifted mathematicians were not routinely exposed to true appropriate-challenge nor supported by teaching and learning methods tailored to this purpose. Gifted mathematicians might therefore feel prematurely confident about their ability (Sanchez & Dunning, 2018) to pursue advanced mathematical-development during their further education, mistaking their history of succeeding in under-challenging activities for the true mathematical confidence (Avhustiuk, Pasichnyk & Kalamazh, 2018) they might enjoy as particularly-capable mathematicians maximising their potential at a higher level.

Maximum potential beyond A-Level grades potentially takes many different forms for gifted mathematicians during further education (2.4.2). However, whether pursuing independent research projects, mathematics competitions, or acceptance at an elite university, the nature of advanced mathematical-development is likely to share a common focus on problem-solving in new and novel ways (Siklos, 2019). In this way, the nature of advanced mathematical-development is shared by gifted mathematicians during their further education with a variety of different ambitions. This is a different process to acquiring knowledge of additional mathematics topics (Chytrý *et al.*, 2020), with which gifted mathematicians are usually proficient. Instead, problem-solving requires gifted mathematicians to develop their thinking metacognitively, which requires creativity (Kozlowski & Chamberlin 2019). It has also been suggested (Kozlowski & Si, 2019) that problem-solving activities afford opportunities for mathematicians to form and evaluate their own solution paths independently. Elgrably and Leikin (2021) highlighted a specific association between an individual's development as a mathematical problem-solver and their improved ability to acquire new mathematical knowledge. Problem-solving has therefore been suggested as a core activity for gifted mathematicians (Singer, Sheffield & Leikin, 2017; Singer *et al.*, 2016). Ngiamsunthorn (2020) noted the particular importance of gifted mathematicians' creative problem-solving abilities during undergraduate studies, and the necessity of developing them throughout Higher Education, as many undergraduate mathematicians arrive under-skilled in this area. Gifted mathematicians with a history of being under-challenged prior to their further education are therefore potentially ignorant to what advanced mathematical-development actually entails, which can cause disconcertion when they begin to pursue it. In particular, where they have previously remained largely concerned with learning to solve specific types of problems, advanced mathematical-development requires them conceive of methods to solve problems independently (Siklos, 2019). The thinking style to be honed throughout advanced mathematical-development is therefore metacognitive in nature (Drigas & Mitsea, 2021) in that it pertains to directly considering the ways methods can be designed when faced with mathematical problems, with the emphasis no longer on the specific method under consideration in that scenario (Sîntămărian & Furdui, 2021). Metacognitive problem-solving is a more-abstract skill (Villani *et al.*, 2019) than a typical gifted mathematician has honed within their mathematical development prior to the further education phase. Many therefore perceive discomfort in suddenly needing to think in a metacognitive way (Halmo, Yamini & Stanton, 2024). Hence, gifted mathematicians may require the development of metacognitive thinking associated with advanced mathematical-development to be scaffolded by a teacher (Matsuda, Weng & Wall, 2020) to ultimately succeed in the endeavour.

The phenomenon of gifted mathematicians struggling when faced with abstract challenge is not unlike the Kruger effect (Kruger & Dunning, 2000), where beginners realise they have overestimated their ability. Such learners are conditioned to believe that all levels of future development will be as straightforward to master as the relative basics encountered previously. Gifted mathematicians therefore often initially overestimate their ability and confidence to engage with the higher-level and more-abstract activities (Lévy-Garboua, Askari & Gazel, 2017) which would constitute appropriate challenge during their further education. In describing gifted mathematicians working effectively at this level, Siklos (2019) suggested that puzzling with enjoyment through complex problems should be a typical developmental-practice, but often proves to be more daunting than pleasant. For the many gifted mathematicians who find this practice alien, this level of metacognitive problem-solving is likely to pose their first ever challenges acquiring mathematical knowledge and skills. Rochayani's (2024) research into gifted mathematicians' success in mathematics competitions (UKMT, 2023, 2024) concurred. Her study (Rochayani, 2024) highlighted that gifted mathematicians need most support in the early stages of advanced mathematical-development to maximise their results in competitions. It was further posited that such learners are commonly unconsciously unaware of the extent of potential areas of mathematical development that are possible during their further education (Siklos, 2019). This might be argued to be a form of unconscious *incompetence* if gifted mathematicians were novices (Bach & Suliková, 2019). However, despite routinely being under-challenged, they typically hold significant mathematical knowledge and skill compared with typically-developing peers. Hence, this phenomenon is perhaps more-accurately described as unconscious *unknowing*. Having developed a certain level of mastery has been described as initially affording them higher mathematical self-efficacy than their peers (Korkmaz, Ilhan & Bardakci, 2018). When gifted mathematicians eventually encounter supra-curricular activities which are significantly-more complex, they might be especially concerned that their underlying belief in their infallible ability to make mathematical progress is even momentarily disrupted (Ronksley-Pavia & Neumann, 2020). This experience undermines their mathematical self-efficacy, subsequently increasing their risk of disengaging from advanced mathematical-development (Ozkal, 2019).

Mofield and Parker Peters' (2018, 2019) studies into traits of the gifted concluded that such people are more likely to be perfectionistic. Perfectionism has, in turn, been identified as a key promoter of positive self-efficacy in gifted people (Akkaya, Dogan & Tosik, 2021). Mathematics as a subject is often interpreted as one in which an individual's conclusions can either be correct or incorrect (Jansen, 2023; Shen *et al.*, 2021; Shinariko *et al.*, 2020; Radmehr & Drake, 2018); even slight mistakes are therefore often interpreted by gifted mathematicians as a divergence from perfection (Alvidrez, Louie & Tchoshanov, 2024; Aziz & Hakim, 2024; Maulyda *et al.*, 2020). Rice and Ray's (2018) intimation that gifted people are more likely to struggle to adjust their mindset after even minor divergences from perfection is therefore particularly worrying for gifted mathematicians during their further education. In arguing for provisions designed to help gifted people overcome perfectionism, Greenspon (2021) described their reliance on feeling infallible as a core weakness. He (*ibid.*) suggested that this is an impossibly-high standard which inevitably causes psychological distress when gifted people failed to meet it. Feeling even momentarily incompetent when encountering conceptual difficulty in advanced mathematical-development has therefore not only been argued to undermine self-efficacy, but also self-worth (Grugan *et al.*, 2021). Becoming enlightened by a first experience of the reality that advanced mathematical-development during their further education might be far larger and more complex than anticipated is therefore potentially a daunting prospect (Kahraman & Bedük, 2016). The further education phase is typically only two years long, which commonly causes learning experiences to be intensified (Keenan & Kadi-Hanifi, 2020; Macfarlane, 2018). This makes the necessity to begin advanced mathematical-development feel more urgent to gifted mathematicians during their further education. Consequently, content which is appropriately-challenging for gifted mathematicians during further education therefore falls short of the escapade it could be (Siklos, 2019), becoming an emotionally-negative and distressing affair (Karpinski *et al.*, 2018; Kennedy & Farley, 2018). This absence of the coping skills which enable resilience suggests that, to be successful in pursuing their ambitions for advanced mathematical-development, gifted mathematicians require structured introductions to these activities, and specialised support to engage with them (Shukla, 2022), during their further education phase. However, previous discussion (2.5) has concluded such support is lacking. It is therefore easier for gifted mathematicians to completely avoid this distress than develop the resilience to rise to the new challenge during further education, meaning many choose to relinquish their original ambitions (Besnoy, Jolly & Manning, 2021). Instead, many seek solace in what they already know they can succeed at: achieving highly in standard (A-Level) studies (van Tassel-Baska, 2018). This is worrying in light of Svendsen and Burner's (2023) finding, that focusing on assessment grades leads to less engagement in learning for gifted people. Moreover, when gifted mathematicians divert their focus in this way, this ultimately means they neglect their opportunity to develop the necessary perseverance and resilience (Taylor, 2009). They are therefore unable to reconstruct their mathematical self-efficacy and start enjoying their advanced mathematical-development in the way it has been suggested they might (Siklos, 2019).

2.5.4 Supporting Positive Mathematical Self-Efficacy Through Developing Resilience and Motivation

The notion of being continually under-challenged in secondary education and associated obstacles in sustaining positive mathematical self-efficacy during the further education phase affects many gifted mathematicians. However, gifted learners who rise to this and similar challenges without significant support have also been observed (Shukla, 2022; Neihart & Yeo, 2018). However, it has been argued that the very concept of appropriate challenge requires gifted learners to actually feel challenged (Fiedler, Nauta & van Henegouwen, 2020). When gifted learners do not actively experience some difficulty, it might therefore be argued that they are not yet being *appropriately* challenged (Zepeda, Martin & Butler, 2020). Barnett (2019) held that gifted learners cannot make meaningful progress in the absence of this feeling. Hence, to truly constitute appropriate challenge, gifted mathematicians who are not currently feeling stretched by higher-level activities require such tasks to be complexified still-further (Özdemir & Isiksal Bostan, 2021a). When activities *are* sufficiently elevated in difficulty, such gifted mathematicians will potentially become exposed to the same issues around sustaining mathematical self-efficacy as their peers, just at a later stage. Subsequent research must therefore take care to ensure that activities planned as part of any trial interventions do constitute true, and hence appropriate, mathematical challenge for its participants. Moreover, it must ensure any theories employed reflect a person-led stance, and hence account for variations in gifted mathematicians' existing competence within their advanced mathematical-development.

In their investigation into gifted students' perspectives of teaching practices in mainstream classrooms, Gomez-Arizaga *et al.* (2020) posited that many such learners viewed typical classroom experiences negatively. However, some of their (*ibid.)* participants were notable exceptions. These gifted students responded positively when teachers utilised new pedagogical practices which specifically attempted to engage gifted learners in mixed-ability classrooms. A top-down approach to ability differentiation is one such practice. The participants in this study (*ibid.*) appreciated the distinct way they were treated in these scenarios compared with more-traditional classroom environments. The authors (*ibid.*) put forward that such pedagogies were uncommon, but not non-existent, in mainstream education. Examples of gifted students who have experienced such practices in regular classrooms and subsequently go on to better maximise their opportunities to progress in gifted programmes have also been reported (Backes, Cowan & Goldhaber, 2021). When gifted mathematicians benefit from tasks designed to offer them appropriate challenge through ability-differentiation in mixed-ability lessons during secondary education (Özdemir & Isiksal Bostan, 2021a, 2021b) they therefore have earlier experiences of higher-level challenge which supports their engagement in advanced mathematical-development upon reaching the further education phase. Thus, such learners better utilise their quality of giftedness to connect them with higher ideas and concepts (Gavin, 2021). This better prepares gifted mathematicians to take part in advanced mathematical-development. Moreover, it fortifies their resilience through previous experiences of successfully navigating upward shifts in difficulty (Worrell *et al.*, 2019). Even limited success in this regard affords these learners with additional defences through which to shield their mathematical self-efficacy. Ahn and Bong (2019) opined that belief in one's future success is a core tenet of ongoing motivation to maximise academic potential. Moreover, Talsma *et al.* (2018) propounded that real experiences of success strongly influence a learner's belief they will succeed in the future in similar endeavours. To address the knowledge gap, research must therefore explore the initial stages of advanced mathematical-development in detail. This will subsequently establish how gifted mathematicians can be supported to begin undertaking advanced mathematical-development, providing them with opportunities to experience success which they can utilise to build motivation and sustain positive mathematical self-efficacy during their further education (Wilson & Janes, 2008).

Starja, Nikolova, and Shyti (2019) reported that some gifted mathematicians are momentarily fazed by embarking on advanced mathematical-development. However, they found that some were better-able to remain resilient and continue despite discomfort when appropriately challenged for the first time. As these gifted mathematicians have a natural tendency to persevere, they create their own opportunities to rise gradually when presented with new challenges until they have established a sense of comfort from which to pursue such development (*ibid.*). They, perhaps implicitly, perceive any doubts that challenging material might summon regarding their ability to make progress as temporary fluctuations to, rather than a longer-standing destruction of, their mathematical self-efficacy (*ibid.*). Gifted mathematicians naturally possessing a positive outlook, who perceive obstacles as opportunities even when such obstacles create a sense of self-doubt (Parish, 2018), might therefore be interpreted as a nuanced means through which such learners sustain confidence in their ability to make mathematical progress. Their self-efficacy as a skilled mathematician is supported by their self-efficacy as a mathematics learner who believes they are fully capable of developing their mathematical knowledge and skills *despite* conceptual difficulty. In this way, mathematical self-efficacy is defended by an individual's belief they can make mathematical progress, not simply a belief that they are skilled at doing mathematics (Mun & Hertzog, 2018). Initially the concepts sound similar. However, acknowledging future challenges and being intrinsically willing to accept them affords those holding the former belief with resilience lacking in those with the latter. They are therefore better prepared for any difficulties that may arise. This preparation facilitates their ongoing motivation throughout advanced mathematical-development which subsequently allows them to become more experienced as mathematical problem-solvers (Khaliq & Rasool, 2019). Having been more likely to have continued with and overcome challenges, and gone on to master a high-level mathematical skill, such learners are also more likely to have experienced the associated feelings of satisfaction and fulfilment (Sriraman, 2021; Petry, 2019). Czarnocha and Baker (2021) described the effects of these moments of mastery after a process of struggle as a positive experience that, once encountered, demonstrate the existence of future rewards which serve as incentives to remain motivated. Perseverance, therefore, might be said to organically transform into intrinsic motivation, which both Knežević, Blanuša, and Hilčenko (2018) and McCoach and Flake (2018) considered a key factor in ensuring gifted people maximise their potential. This underlines the need for research to more-closely explore the mechanisms through which gifted mathematicians can develop perseverance within advanced mathematical-development specifically. Moreover, the study can address a knowledge gap by investigating how teachers can support gifted mathematicians without this perseverance to develop it.

2.6 The Zone of Proximal Development as a Theoretical Framework for Conceptualising Advanced Mathematical-Development

Identifying theories of giftedness to frame the language through which mathematical giftedness and its development during further education can be comprehended was one crucial aim of the earlier sections in this literature review (2.2). In particular, the ‘quality of giftedness’ theory (2.4.1) was utilised to understand gifted mathematicians during their further education in their own right, and how they might experience advanced mathematical-development. The theory was utilised to explore and make sense of a typical experience of a gifted mathematician both prior to and within their further education, and its subsequent or current impact on their mathematical self-efficacy (2.5.3). Consequently, several advantageous features of theoretical frameworks to conceptualise advanced mathematical-development have been identified. As gifted mathematicians are likely to begin the further education phase with a variety of previous experiences (2.5.3), theories needed to be flexible enough to vary to the specific presentation of each individual. Moreover, they were also required to be robust enough to evolve dynamically as each gifted mathematician’s perceptions of advanced mathematical-development potentially ebbed and flowed. It was also noted that the experience of developing mathematical gifts during further education can be very uncomfortable for the gifted mathematician, potentially causing frustration before resilience can be cultivated (2.5.3). Theories therefore also needed to account for the influences of feelings and emotions in human development (Lerner, 2018). This feature also addressed the pleasure experienced when succeeding with difficult tasks, especially those that caused earlier frustration, which created motivation for advanced mathematical-development. One fundamental basis of pedagogical research is that teachers have a significant role to play can elicit improvement in their students. This is particularly applicable to gifted mathematicians, and suitable theories were therefore required to conceptualise the role of other people in a gifted mathematician's perception of advanced mathematical-development. Suitable theories also needed to respect the educational environment and activities they were being applied to (Spangler & Williams, 2019). Metacognitive problem-solving has already been identified as the common pursuit of gifted mathematicians during their further education (2.5.3). Theories to be utilised in the theoretical framework were therefore required to be particularly applicable to problem-solving activities.

The importance of the influences of other people within problem-solving interactions suggested that social learning theories, which conceptualise learning and development as a product of collaboration (Busch & Watson, 2021), were particularly suitable for synthesis into the theoretical framework. Many such theories build on the behaviourist tradition, which puts forward that learning happens predominantly through imitation (Smith, 2020). Hence, this tradition suggests that learners' prolonged exposure to detailed demonstration drives their acquisition of the knowledge or skill being demonstrated (Braddon-Mitchell, 2019). Traditional social learning theories therefore seemingly doom learners never to exceed the knowledge or skills of their teachers, and have faced much criticism on this basis (Bates, 2023; McLay *et al.*, 2018). This criticism is particularly applicable in studies into gifted mathematicians' perceptions of their own experiences; as self-perceptions exist intrinsically (2.5.3), they should not be theoretically limited by others. Additionally, such theories suggest that all learners learn effectively in very similar ways (Aubrey & Riley, 2022; Race, 2020), a further unsatisfactory feature given that gifted mathematicians are anticipated to begin their further education phase with various individualised needs (2.5.3). However, all such contradictions are potentially rectified by contemporary notions of sociality within learning (Proctor & Niemeyer, 2020; Rubtsov, 2020). In particular, the nuances of an individual's cognition can be applied to explore the mechanism through which they deliberately pursue their development, despite this development being the outcome of the interpersonal relationships within their social context (Lin, Chen & Cheung, 2024; Żuromski & Pacholik-Żuromska, 2024; Bakhurst, 2023; Ahn, Hu & Vega, 2020). The subsequent sections in this chapter put forward the ZPD (Vygotsky, 1978), and the notion of More Knowledgeable Others (MKOs), as the predominant theory being applied as a framework for exploring gifted mathematicians' perceptions of advanced mathematical-development.

The ZPD was originally defined as follows:

> *The ZPD is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers* (Vygotsky, 1978, p. 86).

Much subsequent work has carried forward the inherent positivity of this original definition (Barrs & Richmond, 2024; Barrs, 2022; Hedegaard, 1992; Tudge, 1992), which is phrased around what an individual *can* do. In particular, an individual's independent capability is emphasised by placing it first within the definition (Nardo, 2021). Furthermore, an individual's overall capability is not considered diminished by the tasks they require support with; rather, this is phrased as their *potential* (Newman & Latifi, 2021). However, gifted mathematicians commonly perceive themselves negatively when meeting appropriate challenge during their further education (2.5.3). When encountering such tasks for the first time, gifted mathematicians have been described as viewing the tasks as extremely difficult or impossible (2.5.3). They therefore require support from another person with a high level of mathematical knowledge and skill (Darmayanti *et al.*, 2023). Moreover, a negative perception of ability is a limiting factor those who experience it must overcome. Reflecting this negative perception within the conceptualisation of their ZPD is therefore advantageous for considering the earliest stages of their advanced mathematical-development. Moments when gifted mathematicians achieve something they originally believed they were incapable of even with guidance are discussed as particularly motivational (2.5.4). Negative self-perception should not therefore be feared. It holds a power to be transformative for a gifted mathematician's future perceptions of advanced mathematical-development, and therefore deserves its own emphasis. An alternative conceptualization of the ZPD which maximises the theory's usefulness for researching gifted mathematicians' advanced mathematical-development might therefore be put forward as follows: "The cognitive position between what a gifted mathematician currently perceives as unfeasible for them even with support from another person who is highly skilled in the task, and that which they are already capable of achieving unaided."

Within the ZPD, it is theorised that close social interactions between the learner and skilled mentors (MKOs) is crucial in building this bridge between the straightforward and the (currently) unfeasible (McLeod, 2024a, 2024b). Moreover, MKOs facilitate an individual's simultaneous construction and traversal of this bridge by gauging their current ZPD and implementing tailored strategies, known as scaffolding (Puntambekar, 2022; Margolis, 2020), to support their movement forward (Abtahi, 2014). Learners also progress their ZPDs through social interactions with their MKOs more-generally, not just through utilising the scaffolding designed specifically by the MKO (Xi & Lantolf, 2021). The process of tailoring scaffolding effectively has been interpreted as the act of assessing and providing optimal challenge for learners (Kim, Belland & Axelrod, 2018). By the very conception of the ZPD, optimal challenge would not feel easy. Hence, this theory is particularly suitable for conceptualising and analysing an individual gifted mathematician's moments of frustration, and how they work through frustration to build resilience, when exposed to appropriately-challenging content during further education (2.5.3). The role of the MKO, then, is to structure this ongoing process to facilitate the evolution of a learner's ZPD. If considered purely from the viewpoint of mathematical knowledge this would still appear to limit gifted mathematicians to never exceed their MKO in this regard. However, in addition to applying the theory to the acquisition of mathematical knowledge itself, the study also sought to explore each gifted mathematician's ZPD as it related to the development of resilience when encountering appropriately-challenging mathematical content (Turgut & Uğurlu, 2024). Therefore, gifted mathematicians need not exceed their MKO in this manner to develop their mathematical knowledge and skills without such a limit. Abtahi (2018a) made similar interpretations when applying Vygotsky's theory to mathematics education in that she considered the theory to be one of personal development, not just of learning. Eun's (2019) suggestions went somewhat further, propounding that an individual's inner voice and dialogue have a strong influence on their perception of their ZPD. Modernised notions of the theory therefore lend themselves effectively to the exploration of gifted mathematicians perceptions of advanced mathematical-development, which are observed via, and influenced by, their inner dialogue (Carroll, 2022; Putri, 2021; Hunter & Sullins, 2020). Moreover, Belland, Kim, and Hannafin (2013) applied the theory as a means of designing scaffolds which improve motivation alongside cognition, a factor already highlighted as a key driver of gifted mathematicians' progress (2.5.4). This adapted version of the ZPD was therefore applied as the theoretical framework.

Throughout the remainder of the thesis, many references will be made to where a given activity is situated relative to an individual's ZPD. Moreover, there are many similar references to how each individual's ZPD changed over time, or how it might be theoretically anticipated their ZPD should change in light of perspectives in the literature. See, for example, 5.3.3. However, there are many different sets of vocabulary to describe these nuances within the myriad literature utilised to inform the critical evaluation. For consistency, it was therefore necessary to establish language specific to the notion of a gifted mathematician's ZPD. A variety of activities lie inside an individual mathematician's ZPD, depending not only on the specific individual, but also the MKO and their strategies for scaffolding progress (3.9.2), and wider social influences in that particular instance (5.4.5). An individual gifted mathematician's ZPD is not, therefore, comprised of just a single activity. Rather, their ZPD is a neighbourhood of activities that they can be guided around by an MKO, and, after sufficient success with advanced mathematical-development, purposefully navigate of their own accord. This notion of ZPD comprising of a neighbourhood of possibilities inspires an alternative term to within or inside the ZPD. Instead, the term vicinal to the ZPD will be utilised, which means 'belonging to the ZPD's neighbourhood' (Lexico, 2024b). Activities which they are already capable of achieving unaided are then termed as beneath the ZPD, to reflect the idea that becoming accustomed to consistent underchallenge is often what causes a gifted mathematician difficulty when first encountering activities actually vicinal to their ZPD (2.5.3). Likewise, overchallenging activities, which a gifted mathematician currently perceives as unfeasible for them even with support from another person who is highly skilled in the task are termed as beyond the ZPD. This subsequently motivates the verb transcend to describe a gifted mathematician's ZPD which has shifted upwards, that they are now capable of succeeding in activities they previously believed were beyond them. Transcend has several possible opposite verbs (WH, 2024), of which subceed is chosen on the basis it is also conjures a notion of not succeeding. This reflects the idea that gifted mathematicians often judge their performance in mathematical tasks as successes or failures, attaching particularly negative connotations to failures (2.5.3).

2.7 Summary

This chapter has established a gap in the literature for a pedagogical model of support throughout advanced mathematical-development. Vygotsky's (1978) theory of the ZPD has also been presented as the theoretical framework which can be applied to create this understanding and situate it in a scholarly context. Moreover, this chapter has refined the theory, improving upon its general principles to make them more applicable to the study of gifted mathematicians' social and individual development during further education. The Methodology chapter which follows establishes an onto-epistemic philosophy to design methods for selecting gifted mathematicians to participate and effectively generate the data that the developed theoretical framework was ultimately applied to.

3 Methodology: Uncovering Perceptions of Advanced Mathematical-Development to Create a Model of Supportive Pedagogy

3.1 Introduction

This chapter begins by forming the research questions and objectives (3.2) in light of the literature review and theoretical framework. This establishes the context for the remainder of the chapter, which explains how answers to the research questions were subsequently pursued. Reflexivity was a core ongoing consideration, and the approach to introducing and challenging my reflexive position throughout the thesis is therefore next outlined (3.3). This enables that the role of reflexivity in relation to the study's underlying philosophy to be clarified (3.4, 3.5), and reflexivity considerations to be included throughout the subsequent evaluation of methods and approaches where necessary. The methods of the study are then exposited in a manner consistent with an effective approach to presenting a mathematical argument, by first summarising the study before finer details are explored. This summary (3.6) details the who, what, and when of the study by briefly chronicling its narrative. The subsequent sections then detail the methods employed, justifying the choices made. In particular, the tailoring of digital diary-interview method for use with gifted mathematicians during their further education is exposited (3.10), and the subsequent refinements to the data-analysis procedure explained and defended (3.12).

3.2 Research Objectives and Questions

The identified literature gap pertains to the need for an improved understanding of advanced mathematical-development and the subsequent development of a pedagogical model (6.3) for supporting gifted mathematicians to effectively pursue it. The literature review arrived at this conclusion through identifying why many gifted mathematicians forgo their higher ambitions during their further education. In particular, the resilience to bounce back when challenge feels uncomfortable breeds a confidence to pursue advanced mathematical-development which sets up gifted mathematicians with this quality to succeed (2.5.4). This subsequently improves their chances of fulfilling their potential. The research must therefore: seek to understand how gifted mathematicians nurture resilience and motivation within advanced mathematical-development; determine the factors that influence this, either positively or negatively, and the extent of their influence; and investigate and design pedagogies that both emphasise the positive factors and diminish the negative ones to better facilitate gifted mathematicians' achievement of their maximum potential during their further education. The research questions can therefore be stated as follows:

1. How do gifted mathematicians perceive their experiences of advanced mathematical-development throughout the further education phase?
2. What implications do gifted mathematicians' perceptions of advanced mathematical-development have for effective pedagogical approaches which support them through the challenges they associate with this experience?

3.3 Approach to Acknowledging Reflexivity Considerations

Researchers investigating their own practice are intertwined with their projects, making it essential to explicitly consider how their preconceptions influence the research process (Creaton, 2020). This was especially true in my study given my personal experiences as a gifted mathematician (1.5) which further embedded my preconceptions of what effective provision should entail for such learners during their further education. While it is acknowledged that many qualitative researchers attempt to explore the influences of their preconceptions reflexively by attempting to bracket them (Hoskins, 2020; Dodgson, 2019), I approached this differently. In particular, my preconceptions were so prevalent they required explicit ongoing mention and challenge to ensure their influence was adequately considered (Pihkala & Karasti, 2024). This helped overcome the associated concern that bracketing is a particularly limited approach to addressing reflexivity considerations in studies with a significant element of research-researcher intertwinement (Gregory, 2019). Reflexivity considerations are therefore included when warranted through the remainder of the thesis.

For the present purposes, it suffices to clarify how my position might have influenced my choices around establishing an appropriate onto-epistemic philosophy for my study. Gifted mathematicians have been described as having thoughts which align with the logic of mathematics itself (2.4.2). An early reflection, therefore, was that I would be particularly minded towards positivist onto-epistemologies. However, this was not entirely consistent with the emphasis on perceptions at the heart of the study. Participants were valued for their detailed individual opinions. Such opinions are by their very nature subjective. Care was therefore needed on my part to ensure I was fully invested in a qualitative approach to my study and was actively embracing the power of subjectivity within an interpretivist paradigm to create meaningful knowledge.

3.4 Onto-Epistemic Position: Gifted Mathematicians' Phenomenological Perceptions

A study's ontology is an understanding of what data exists that might offer answers to its research questions (Grix, 2019; Kelly, Dowling & Millar, 2018). Giftedness researchers have considered giftedness as an evolving quality, present throughout a lifetime, and specific to each individual (Freeman, 1998). Hence, many researchers take a view that giftedness is experienced independently, and best described by the person experiencing it (Gomez-Arizaga *et al.*, 2020; Erdogan & Yemenli, 2019). Consequently, many studies adopt phenomenological ontologies (Smedsrud, 2018; Mullet, Kettler & Sabatini, 2017; Price *et al.*, 2016). Phenomenology puts forward that experiences happen with people (Paley, 2018), placing individuals at the centre of these experiences (Zahavi, 2019). Hence, phenomenology conceptualises individuals as the primary mediators of the wider understanding of these experiences (Beck, 2020; Schmitz, 2019). Phenomenological inquiries therefore view individuals' perspectives as apposite data for illuminating the experience under investigation (Sundler *et al.*, 2019; Zahavi & Martiny, 2019). Moreover, phenomenological inquiry naturally aligns with the study's intention to give gifted mathematicians a platform (1.2), emphasising their individual views and perceptions (Neubauer, Witkop & Varpio, 2019; Wojnar & Swanson, 2007). Finally, mathematical giftedness during the further education phase has been identified as presenting in a variety of ways

(Simensen & Olsen, 2024), specific to each gifted mathematician (2.4.2). Respecting their individuality was therefore an important aspect of creating a pedagogical model applicable to the further education of a variety of gifted mathematicians.

Phenomenological data were sought was to inform the design of a pedagogical model to support gifted mathematicians through the challenging elements of advanced mathematical-development that the study subsequently identified. Valuing individuality was one important factor in the study. However, the model was intended to be the basis for future practice and research with gifted mathematicians across the FE sector more widely (Maxwell, 2019). Consequently, the way the data were utilised to construct this knowledge needed to be considered valid and reliable to practitioners, in addition to a doctoral thesis' academic audience. A study's epistemology is its philosophy towards how its ontological position is operationalised to provide reliable answers (Kotzee, 2019). An appropriate epistemology therefore needed to be capable of contextualising individual views against each other, mediating them collectively to make robust recommendations for practice. Phenomenological data were therefore evaluated within an interpretivist paradigm (Kumatongo & Muzata, 2021; Dean, 2018), seeking to create meaning by interpreting data from a variety of perspectives (Norwich, 2020). Unfettered interpretivism has been criticised in phenomenological studies for valuing third party subjectivities over that of each individual participant (Blaikie & Priest, 2017, 2019; Patton, 2019). Such arguments purport that third-party interpretations are a form of disrespect (Gros, 2017), believing the individual's interpretation should transcend all other perspectives. Transcendence is also viewed as a means of protecting a study from bias by, for example, restricting a researcher's latitude to adopt favourable interpretative positions (Pham, 2018; Primus, 2009). However, hermeneutic variations of phenomenology balance the need to preserve individuals' perspectives while interpreting them collectively (Nigar, 2020; Suddick *et al.*, 2020). A core focus on using a collective interpretative lens for the good of the group being researched counters the discussed criticisms around researcher bias (Schmitz, 2019). Instead, such subjectivity and interpretation are particularly valued in practitioner research. Teachers are well placed to explore classroom issues and investigate potential solutions (Maguire, 2019). In particular, as an experienced teacher of gifted mathematicians across a variety of settings (1.5), I was uniquely placed to analyse gifted mathematicians' views by ultimately developing a collective interpretivist lens. My perspective on these issues held a power to serve as the necessary link between the study's participants and other gifted mathematicians more broadly, allowing the findings to be adapted to and applied in a variety of educational settings (Rapley, 2018).

3.5 Phenomenography with Gifted Mathematicians

A final consideration influencing the choice of methods is how the onto-epistemic position is operationalised to generate and analyse data in a valid and reliable way (Hayashi, Abib & Hoppen, 2019; Vakili & Jahangiri, 2018; Mohajan, 2017; Noble & Smith, 2015). The plethora of meanings and values associated with validity and reliability complicates this process in qualitative research (St-Onge *et al.*, 2017). It is therefore not as straightforward to simply label qualitative studies as reliable, unreliable, valid, or invalid. Instead, each choice within the research should be explicitly debated to detail limitations transparently, judging their strengths and weaknesses with respect to the established onto-epistemic ideals (Stinson, 2020). This process is aided through developing methodological principles which inform the choice of methods (Khatri, 2020). It has been repeatedly noted that gifted mathematicians' value was as individuals with rich stories to tell (3.4, 4.2, 4.3, 4.4). This therefore suggested that the developed methods needed to respect the individual nature of gifted mathematicians' phenomenological data, facilitating every individual's expression of this in a clear, detailed, and chronological way (Hennink, Hutter & Bailey, 2020). Moreover, the methods also had to ensure enough context was collected to reconstruct how each person perceived the experience accurately (Bell, Waters & Johnson, 2024), and include opportunities for them to directly guide this reconstruction (Denzin *et al.*, 2023). One methodological philosophy which supports participants to offer this context is phenomeno*graphy*, where participants take the lead within the process of contributing their perspectives as data (Orgill, 2012). This often includes the ability to do so at any time of their choosing, to capture the minutiae of their evolving experience without direct interaction with the researcher (Hajar, 2020). Such methodologies therefore task their participants with an atypically-high level of autonomy, further enabling their individuality to be naturally present in their phenomenological data through minimising researcher interference (Washburn, 2018). Thus, data are generated autobiographically, capturing detailed information about how participants feel (Creswell & Creswell, 2022) at various stages of an inquiry in addition to their direct perspectives on the phenomenon under investigation (Åkerlind, 2018). Stolz (2020) critiqued this participant autonomy as a handover of responsibility within phenomenographical educational research, viewing it as a potential departure from the phenomenological principles. Such principles strengthen a study's validity (*ibid.*). In particular, when data are contributed outside of an interaction with the researcher, the researcher forgoes any immediate opportunity to seek clarification which facilitates a better understanding of an individual's perspective. It was therefore imperative that participants' opportunities to provide phenomenological data autobiographically were structured in ways which facilitated the divulgence of detailed information (Blaikie & Priest, 2019). This balanced the need to respect their individuality against the objective of eliciting relevant insights.

3.6 Summary of the Exploration of Advanced Mathematical-Development

Now that a phenomenographical approach (3.5) has been identified, the narrative of the study can be explained and justified in relation to how it was guided by the underlying onto-epistemic philosophy (3.4). To do so, a summary of the study is first presented below. The language used to refer to various stages of the study will be made explicit. Moreover, the summary contextualises the critical evaluation of methodological literature that follows in the chapter's remaining subsections, which establish the justification for the research design.

Three gifted mathematicians in Year 12 were invited to take part (3.8) in regular problem-solving sessions intended to immerse them in advanced mathematical-development to enable their perceptions of this experience to be explored. The sessions involved working on mathematics problems at a level of appropriate-challenge (3.9.1) exceeding the demands of A-Level study. This is an experience that earlier discussion (2.5.3) determined potentially undermines their success. The participants were also encouraged to undertake problem-solving activities in their own time. However, the sessions served as opportunities for me to offer them specific support with problem-solving and work together with them on the problems (3.9.2). Moreover, eight of the sessions were of particular methodological importance when participants were recording their views. These sessions were sufficiently spaced to allow advanced mathematical-development to take place, and hence for participants to have offered perspectives pertaining to both its initial stages and at a later point in time. It was theorised the participants would find the earliest times in this experience the most challenging (2.5.3). The sessions therefore began in January 2023, which was chosen as the earliest point in the academic year that the participants could engage with this level of challenge having learned a sufficient amount of A-Level Mathematics (OCR, 2024b). Participants recorded diary entries throughout a two-week period during which four problem-solving sessions took place. As diaries were employed as the primary research instrument (3.10.1), this phase of the study is henceforth referred to as Diary Phase One. Sessions were then held regularly throughout the academic year. The participants subsequently kept diaries again in June 2023, known as Diary Phase Two, at which point they had lived through the early challenges of advanced mathematical-development and were in various stages of overcoming those challenges.

To closely explore the perceptions of each individual within this evolving experience, digital diaries were developed as the first method of inquiry. The participants made diary entries using iPads to facilitate a variety of formats (3.10.1). There were specific moments when they were asked to make entries (3.10.1), in particular around the times we met for problem-solving sessions. They also had ongoing access to their diary and could make additional entries whenever they decided to (3.10.1). The diaries had some specific questions, checklists, and prompts to support the participants to think about the specific factors which had influenced their perceptions of advanced mathematical-development positively or negatively (3.10.1). The same diary format was utilised in Diary Phases One and Two. The participants continued to pursue problem-solving activities regularly when they returned to school for Year 13 in September 2023. Their involvement in the study concluded with interviews tailored specifically to each of them, which took place in December 2023 and January 2024. This was their opportunity to reflect on the entire experience retrospectively and provide further insights which triangulated their earlier perspectives (3.10.2), and guide the subsequent data analysis (3.11, 3.12). Appendix Two presents the research timeline.

3.7 Ethics: Gifted Mathematician as Practitioner and Researcher

All the participants were aged 16-19. The study therefore included minors, making it necessary to submit a full ethical review. The ethical review approvals are provided in Appendix Three.

As participants were my own students, I held power over them not shared by independent researchers (Temple, 2019). Teacher-researcher duality has posed particular challenges around ensuring data validity (Clark *et al.*, 2020). My students could have felt coerced to participate (McNiff, 2017), fearing repercussions such as restricted access to enrichment activities should they decline. Invitations to take part (Appendix Four) therefore included a precise overview of the activities involved, and repeated reassurance that not taking part would not affect a student's daily educational experience (Kirby, 2020). The information sheets (Appendix Four) therefore included: a comprehensive overview of why the research was taking place and what it involved; its benefits for gifted mathematicians (Ngozwana, 2018); any potential risks to participants (Haider, 2022); and reassurance they would not be treated differently based on whether or not they took part (Iurea, 2018). Information sheets were tailored separately for participants and their parents/guardians to ensure informed consent (Shirley *et al.*, 2021; Ferreira & Serpa, 2018). Participants were also routinely reminded they could withdraw at any time until the specified deadline to continually reaffirm their consent (Barrow, Brannan & Khandhar, 2022).

Those who subsequently participated might have also felt coerced into particular responses. Fearing their negative views could be perceived by me as unwelcome criticisms (Kara, 2018), participants might have felt unable to offer their unfiltered opinion. Moreover, participants might have attempted to judge the type of information I was looking for (Shaw *et al.*, 2019), seeking to offer responses I approved of. Any associated doctoring of their honest opinion could have invalidated the subsequent findings (Rutherford-Hemming, 2018). Coercion was therefore addressed as a core ethical consideration (BERA, 2018; SU, 2019). To mitigate the risk of participants responding in favourable ways, the research questions were withheld from participants until the study's conclusion (Tai, 2012; Groenewald, 2004). Furthermore, my desire to encourage their candour as individuals was emphasised at all stages, so that any influence I was exerting could be utilised to nurture their honesty.

3.8 Sample and Sampling Strategy: Gifted Mathematicians with Relevant Perceptions

As the earliest stages of advanced mathematical-development were to be researched, it was necessary to only invite participants who were unfamiliar with higher-level mathematical problem-solving activities (3.9). The sampling frame therefore only included Year 12 students who were enrolled at the specialist mathematics school (1.5). However, the literature review set out that their individual differences would be important (2.4.2, 2.5.3). Hence, the sample needed to faithfully represent a collection of gifted mathematicians during their further education. Participants were therefore sampled purposively, ensuring all met specific criteria (developed below) (Campbell *et al.*, 2020) so that their insights retained some potential to be applied to other elite mathematical learners (2.4.2) during their further education who share these characteristics (6.4.2; Moser & Korstjens, 2018; Schreier, 2018). Purposive sampling is widespread in giftedness research (Olamafar *et al.*, 2023; Ozlem, Okan & Bilge, 2020; Naif, 2019). However, there is no widely-agreed procedure for identifying giftedness in an individual (Wechsler, Blumen & Bendelman, 2018), nor in a specific subject of giftedness (Maker, 2020). Hodges *et al.* (2018) lamented that identification practices are often heavily reliant on professional judgement. Such practices are often imprecise and influenced by the social context practitioners are teaching within (Lo *et al.*, 2019; Parekh, Brown & Robson, 2018; Tourón & Freeman, 2018). Specific inclusion criteria (Patino & Ferreira, 2018) were therefore developed to objectify this process in the study. It has been recently reaffirmed that considering a variety of indicators objectifies the identification of mathematical giftedness (Nolte, 2024).

Including consistent high-attainment as an indicator of giftedness is common in related research (Worrell *et al.*, 2019; Leikin *et al.*, 2017), but has proved problematic in isolation (Desmet & Pereira, 2021; Hughes, Rollins & Coleman, 2021; Dada & Akpan, 2019), as discussed in 2.4.1. Additional criteria were therefore necessary. In particular, gifted mathematicians have been routinely observed seeking out appropriate challenge and volunteering to take part in such activities (Zavala Berbena & de la Torre García, 2021; Singer *et al.*, 2016). Consequently, all students were told about the research. None were subsequently pursued, and only those actively stating they would like to participate were provided with further information. Their record of taking part in other optional activities in school complemented their mathematics attainment to ultimately select a sample of three gifted mathematicians.

The inquiry was originally planned to take place over a twelve-month period, and hence it was decided to investigate a single cohort’s individual experiences in depth. This made it imperative the right number of participants were included from the beginning (Blaikie, 2018). Determining sample size sufficiency prior to commencement is often described as an inexact process in (Sim *et al.*, 2018).

However, my research benefitted from the outcomes of its pilot in this regard. At the pilot study stage, between five and ten participants were estimated based on outcomes of studies sharing content and methodological similarities (Thompson, 2023; Johnson, Walther & Medley, 2018; van Rijnsoever, 2017; Price *et al.*, 2016). The pilot demonstrated that each participant had the potential to contribute significantly more data than anticipated, and in a greater range of possible formats. As each participant in the later study would contribute data over a longer period it was necessary to be conservative over sample size to ensure the available time for data analysis, a limited resource in the project, was sufficient. The need to explore peer interactions was a further sample size consideration. Scheduling restraints meant that not every participant would be involved in every interaction; there were times when some participants would undertake some activities independently of each other (3.6). It was therefore important to consider how peer interaction would become evident in these sub-coteries of the whole sample. A subgroup of three was considered ideal for this purpose, as, within itself, there were seven distinct combinations of participants. Compared with just three combinations for a group of two, and fifteen for a group of four (Johnson, 2022), a sample size of three offered sufficient opportunities for subgrouping in the later analysis without becoming logistically overwhelming. It was therefore decided to engage three participants in total.

The final sample consisted entirely of male students with no identified special educational needs or disabilities. This posed limitations on the applicability of the findings to females and neurodivergent people, which are further explored in the conclusion (6.4.2). To underline that their views pertained only to neurotypical male perspectives, the participants were pseudonymised using traditionally-male names: Confur, Derwyn, and Ethan. See 4.4.7.2 for further detail of the approach to assigning pseudonyms. The sample did lack some characteristics. However, it is noteworthy that it included people of ages between 16 and 19. The full scope of ages in FE institutions was therefore represented. Moreover, there were participants from a variety of ethnic backgrounds, a range of secondary institutions, and who had previously lived internationally.

3.9 Exploring and Supporting Advanced Mathematical-Development

3.9.1 Advanced Mathematical-Problems

Appropriately-challenging mathematical problem-solving is at the core of gifted mathematicians' advanced mathematical-development (2.5.3, 2.5.4). Szabo, Tillnert, and Mattsson (2024) reported that gifted mathematicians often require support to fully articulate their mathematical reasoning when solving advanced problems in a small group environment. Group problem-solving activities were therefore excellent opportunities to explore advanced mathematical-development in a social context (2.5.4). Hence, the activities in the study contained problems of this nature. Participants and I worked on them collaboratively, and participants often worked on them independently or in groups that did not involve me. In particular, I frequently utilised Sixth Term Examination Paper (STEP) problems (OCR, 2024c). As the admissions test utilised by many universities with highly-competitive entry to degree programmes in the mathematical sciences, STEP is designed to differentiate between students achieving A* grades at A-Level. To do so, STEP tests a mathematician's ability to apply their existing mathematical knowledge and skills in creative, novel, and unfamiliar ways (Siklos, 2019), often combining topics that would be treated separately at A-Level. Succeeding with STEP therefore requires the development of a flexibility of rigorous thinking, a problem-solving skill, rather than the acquisition of additional mathematical knowledge. This makes STEP questions highly suitable for developing problem-solving skills beyond A-Level whilst only having familiar topics as prerequisite knowledge. Moreover, hundreds of these questions exist and are freely available, as are hints, solutions and mark schemes (OCR, 2024c). Hence, not only is this resource accessible to a typical mathematics teacher in an FE setting, but so too are a variety of scaffolding strategies to aid teachers who are not familiar with supporting this development.

The Year 12 students who participated were quite early into their learning of A-Level topics at the onset of the study. Hence, the plan to utilise admissions assessment questions as the basis for advanced problem-solving (3.9.1) required some care. In particular, the questions were audited giving consideration to the A-Level specification topics (OCR, 2024b) the participants had already encountered, thus ensuring the selected questions were suitable relative to the participants' current level of mathematical knowledge. Appendix Five outlines this process in greater detail and presents some of the questions actually utilised in the study.

3.9.2 Approach to Scaffolding Within the Sessions

It was anticipated that the participants would go through a period of adjustment when they began working on appropriately-challenging problems (2.5.3). Hence, the strategies through which they were supported would need to evolve over time in order to support them in the most effective way at the various stages of their advanced mathematical-development. The problem-solving nature of the intervention sessions (3.9.1) meant that inquiry approaches to teaching and learning were apposite (Dorier & Maass, 2020). Reflecting on inquiry based learning approaches led to a variety of scaffolding strategies being investigated (Wrightsmant, Swartz & Warshauer, 2023; NRICH, 2021; Szabo *et al.*, 2020; GMI, 2019; Khong, Saito & Gillies, 2019) in the pilot study (Thompson, 2023), which developed the following hierarchy. Each subsequent level reduces the extent to which the teacher offers direct assistance.

1. Modelling problem-solving processes through examples,
2. Collaborating with learners on the problems,
3. Asking questions to guide students to possible methods,
4. Providing hints to offer subtle guidance to methods, and
5. Giving extended time to create opportunities to overcome obstacles unaided.

The pilot study found that, in general, a reasonable approach to applying this hierarchy is to begin with strategy 4, and then gradually increase the intensity of the support as needed until the right level for the individual was identified. Strategy 5 was reserved for gifted mathematicians who had adjusted to the higher level of rigour and were already showing signs of working independently. The ultimate inquiry-learning approach therefore developed the perspective of Khan (2022) in that I acted as a participator in the problem-solving to support each participant's active engagement through taking on non-traditional classroom roles (Sichangi *et al.*, 2024). However, I also directed learning in the traditional way when an individual required specific help, particularly during the first diary phase. The two paragraphs that follow detail how the above hierarchy of scaffolding strategies was applied differently during each diary phase to achieve an inquiry-learning approach.

Appropriate strategies from the hierarchy needed to be more intense in order to support gifted mathematicians through the early period of adjustment. Specifically, the approach at this time was to introduce a specific problem and give no more than five minutes for the participants to begin thinking about how to solve it. During this time, I assessed them formatively through brief conversations with each individual. Specific participants were then asked to share what they had been able to achieve, and I subsequently summarised it. I also detailed supplementary information regarding how I might have gone about conceiving of or designing a method. I then explicitly implemented the method, presenting the relevant lines of algebra in full detail. Finally, the gifted mathematicians were given more time to further develop the ideas in later parts of the question. I still talked with each individual about their progress, utilising the above approach which started with strategy 4 before gradually choosing and applying more-intense scaffolding strategies (1-3) as needed.

By Diary Phase Two, the scaffolding techniques had largely moved on from the earlier approach of using explicit demonstration and tailored individual interactions as the predominant method of support. In its place, less-intensive techniques from the hierarchy, which allowed greater opportunities for each participant's independent development, became more common. To facilitate them, several possible questions the participants could choose from in the problem-solving sessions were curated, rather than setting a specific problem prescriptively (3.9.1). Participants were also given more time to consider methods and approaches before being given direct help. If they needed support to divine a suitable method, they were asked to state what they were thinking. If from their response it became evident that they were close to success, they were simply reassured that they were doing the right thing, and to keep working on it without additional help initially. When they really did need more support, they were instead asked a specific question intended to help them focus their thinking. This was then supplemented by hints with more detail about the principles and ideas if necessary. Only after these approaches were exhausted were participants collaborated with more actively through detailing the first few lines of algebra or demonstrating part of the solution explicitly.

3.10 Digital Diary-Interview Method: Soliciting Gifted Mathematicians' Perceptions

3.10.1 Digital Diary Method with Gifted Mathematicians

Given the need for spontaneity (3.5), and potential for capturing data in multimedia formats (Spence, 2019), over an extended period (Filep *et al.*, 2017), and at a variety of times (Hyers & Salmons, 2018), *digital* diaries were considered apposite, and hence utilised, as the primary data generation tool. A digital format was created using Microsoft OneNote (Spence, 2019). This was interacted with by participants through an iPad which they kept in their possession throughout the study. Participants included photographs, screenshots, screen recordings, videos (Williamson *et al.*, 2015) and voice notes (Nassauer & Legewie, 2018), and direct electronic handwritten annotation, in addition to contributing data through more-typical typeset formats. See Appendix Six for examples. Having this variety enabled the gifted mathematicians participating to expatiate through their preferred medium in a given moment. For example, it was common for participants to provide an electronically annotated or photographed solution or partial attempt to one of the mathematical problems, accompanied by contextualising data in another form. Such data further explored their perception of what they experienced while tackling that advanced mathematical-problem. Moreover, most contributed to their diaries not only during planned activities with the researcher, but also at impromptu moments. They therefore took advantage of many ongoing opportunities to contribute without direct researcher observation (Bartlett & Milligan, 2020).

The positives associated with allowing the participants more latitude to decide how to respond in a variety of media and at times of their choosing required balancing against the need to generate relevant data. Diaries were therefore designed to include two different formats. In addition to unstructured spaces participants could access and add to at any time, specific pages with some formal structure were included to be completed during the problem-solving sessions throughout Diary Phases One and Two. The structure comprised prompts and checklists (Janssens *et al.*, 2018) which helped participants consider their experiences at important times within the problem-solving activities, namely the moments they thrived or struggled. Subsequently, this facilitated data being generated regarding the specific factors under investigation (2.5) which impacted each participant's perceptions of advanced mathematical-development. One significant change to the diaries following the pilot study (Thompson, 2023) concerned the use of questions within the structured spaces. Pilot participants reported that their ability to freely elaborate was interrupted by questions with multiple parts, hindering their ability to contribute data phenomenograhically (3.5). The diaries were therefore updated following the pilot study (Malmqvist *et al.*, 2019; Ismail, Kinchin & Edwards, 2018). Questions with multiple parts were reformatted as additional prompts and checklists. This allowed participants to contribute data regarding their experiences in the way that felt most natural for them. The diary's contents can be reviewed in Appendix Six.

The use of diary keeping as a data-generation method was not without disadvantages. Primarily, such an approach relies heavily on the participants' commitment to detailing their experiences thoroughly throughout the entirety of a study. Participants in similar diary studies have been observed finding this commitment challenging to sustain (Ramadhanti *et al.*, 2020). Data contributed during the latter stages of such research has been described as less rich than in the former stages (*ibid.*). In my research, participants were only required to keep diaries over two-week periods, and for a maximum of four weeks in total over six months (3.6, Appendix 2). However, the cited project (*ibid.*) was over a longer period. A period of fourteen days is therefore considered an early stage within this phenomenon. This 'tail-off' effect has also been noted as particularly problematic when diaries are utilised to capture information about a wide variety of aspects of participants' lives (Cao & Henderson, 2020). However, in my research participants were only asked to write about how the problem-solving activities they took part in, and their interactions with each other and me within those activities, affected their perceptions of advanced mathematical-development. They were therefore only required to provide specific information in relation to a small set of activities in their daily academic lives. Moreover, time for participants to update their diaries was planned into the problem-solving activities they worked on. The planned occasions when participants were expected to record diary entries helped them establish positive attitudes towards consistently providing rich data within their diaries (Lavy & Eshet, 2018). These opportunities also took place within their usual timetable. Therefore, the quantity of time participants needed to update their diaries outside of regular academic activities was reduced, further supporting their commitment to thoughtfully respond at all stages of the study (Hyers & Salmons, 2018).

3.10.2 Diary-Interview Method with Gifted Mathematicians

Digital diaries provided many advantages for supporting gifted mathematicians to provide valid data. However, there were still some onto-epistemic considerations (3.4) that required addressing. In particular, diaries did not, when taken in isolation, always provide sufficient context to analyse all data within them initially from each individual participant's phenomenological perspective. Moreover, the generated data were extensive, and hence open to a wide variety of interpretations which might not have aligned with each individual participant's intended meaning. This was a particular concern for multimedia and non-worded data (Appendix Six), which sometimes lacked sufficient additional explanation to accurately interpret. It was therefore necessary to include a means of clarifying diary entries, and the analytical lens I was applying, before the initial analysis of an individual's data (4.2, 4.3, 4.4) was concluded. To provide this clarification, interviews were scheduled several months after Diary Phase Two concluded. Participants therefore took part in member checking via the medium of semi structured interviews (Roulston & Choi, 2018), where they were invited to comment on how their data were being interpreted to ensure they were a formative part of this stage of the analysis. Member checking has been considered especially validating in qualitative studies (Birt *et al.*, 2016) when conducted in this way, where the process includes opportunities for participant reflection that subsequently enriches their interpretation of their experiences (Candela, 2019). This was of particular importance in my study, where gifted mathematicians were theorised as potentially benefitting from something they experienced negatively in the moment, only feeling the positives of this experience in hindsight (2.5.4, 2.6). Structured sections of the interviews supplemented diary data with the required clarity and guided the interpretative lens that had been deemed onto-epistemically necessary (3.4). Unstructured sections enabled participants to lead the conversation to provide new insights (Magaldi & Berler, 2020) borne of their reflection since the diaries were completed.

Diaries and semi-structured interviews are both tools which have been extensively utilised. Hence, their validity and reliability for phenomenological inquiry have been established in the wider literature (Creswell & Creswell, 2022; Walliman, 2022; Stolz, 2020; Paley, 2018). However, it remained to ensure their *joint* validity (Denzin *et al.*, 2023) within my specific project. Triangulation has been used to establish reliability through explicitly highlighting findings that arose in multiple ways (Rofiah & Bungin, 2021; Ramsook, 2018). The strength of a particular methodological union is further established by considering the complementarity of the tools being triangulated. Research instruments are particularly harmonious when each provides insights the other tools do not, or enables the insights from other tools to be considered in substantively different ways (Berkeley, 2022). This helps each to overcome their usual limitations (Arias Valencia, 2022; Maxwell, 2022; McCrudden, Marchand & Schutz, 2021). Semi-structured interviews were specifically chosen to support digital diaries with overcoming some of their drawbacks (4.4.5.2). It still remained to explore how diaries enriched the interviews in a similar manner (Dowling, Lloyd & Suchet-Pearson, 2015). A prevalent criticism of semi-structured interviews is their reliance on a participant's memory (King, Horrocks & Brooks, 2019). As my participants were able to refer to their diaries when they needed to, their memory was aided. Moreover, proponents of diary-interview method have contended that diary-keeping facilitates longer term understanding (Salazar, 2024; Zimmerman & Wieder, 1977), hence ensuring the subsequent interviews were the particularly reflective experience they were intended to be (3.10.2). The interview schedule template can be reviewed in Appendix Six.

3.11 Data Processing and Coding Strategy

As participants recorded their views in multimedia formats throughout their diaries (Appendix Six), it was first essential to transcribe and re-record such data as text when needed. This included typing up written data, transcribing what participants said in their audio and video data, and creating brief descriptions of videos' visual content to contextualise them. Interviews subsequently took place after an initial analysis of diary data (Appendix 2). Interview data were initially in an audio format. Recordings were transcribed electronically. I audited the generated transcriptions by comparing them with the recordings as I was familiarising to ensure verbatim records. The entirety of each participant's data then existed in a textual format (Peräkylä & Ruusuvuori, 2018), aiding subsequent analysis.

The coding and theme-development strategies mirrored those of a typical interpretative phenomenological analysis. They were inductive in nature in that categories and themes were created from the data rather than being introduced artificially based on preconceived notions (Vicary & Ferguson, 2024). Theme creation was facilitated by NVivo (Jackson & Bazeley, 2019). In vivo coding, utilising participants own words (Adu, 2019) as codes which represent distinct ideas within their individual data (Elliott, 2018), helped ensure the individualised lenses were intrinsic (Bergin, 2018; Stuckey, 2018) in the phenomenological analyses. As it was anticipated that longer clarifying paragraphs would be attached to any non-worded data (3.10.1, Appendix Six), holistic approaches to coding were utilised. These approaches assigned meaning to lengthier prose where necessary (Miles, Hubermann & Saldaña, 2019) rather than focusing on individual words or short phrases (Linneberg & Korsgaard, 2019). Such passages often alluded to many ideas. Simultaneous coding was therefore utilised, which enabled multiple codes to be attached to words and phrases as necessary (Nowell *et al.*, 2017) to reflect intrinsic links between the ideas participants detailed (Onwuegbuzie, Frels & Hwang, 2016). Ideas that were initially interpreted as holistic codes sometimes became categories in their own right. When this happened, subcodes were created, and consideration given to whether to promote the code to a category. In this way, finer distinctions were drawn between broader ideas which had originally been analysed collectively (Saldaña, 2021). This aided the development of themes by more-accurately representing the extent to which each idea was coded within the data (Williams & Moser, 2019).

3.12 Interpretative Chrono-Phenomenological Analysis Approach

Once all the data were collected, they were grouped together in several distinct ways (described below) to facilitate an analytical approach using the broad principles of interpretative phenomenological analysis as a guide (Smith, 2017). A phenomenological ontology places value on the perspectives of individuals understood in their own right (3.4; Latham, 2024). This is typically respected in the data analysis procedure in interpretative phenomenological analyses by first analysing data separately and thoroughly for each participant (Squires, 2023; Cibotaru, 2022; Cuthbertson, Robb & Blair, 2020) to create themes relevant only to that person (UoA, 2024), henceforth referred to as “individual emergent-themes”. In my research, each participant's data were further subdivided to facilitate an analysis of that individual's perceptions of advanced mathematical-development at distinct times. This is subsequently referred to as chrono-phenomenological analysis, and its justification is the subject of the remainder of this section. The word phenomenological is still employed in relation to the ontological nature of the data (3.4). The chrono prefix is only utilised to reference the specific analytical procedure employed in the study.

Through the digital diaries, data were generated specific to how gifted mathematicians perceived advanced mathematical-development at two distinct stages, with an emphasis on how they perceived it at that moment in time. Moreover, the interviews were designed to give the participants opportunities to reflect on how they perceived their earlier experiences retrospectively. Ajjawi *et al.* (2024) distinguished the two types of phenomenological data as lived experience and hindsight reflection. This distinction was especially important in the study, becoming a tool for better understanding the evolution of perceptions of advanced mathematical-development. It was therefore anticipated that participants' diary data would often pertain to their lived perception at the moment said data were recorded, and that interview data would include a more-significant element of clarification around earlier experiences. However, clarifying data also sometimes appeared in diary entries. The chrono-phenomenological analysis for each gifted mathematician therefore began by isolating all data generated during Diary Phase One, and data generated at later times pertaining to how the individual perceived their lived experiences during Diary Phase One. Data belonging to this subgroup were analysed to create a collection of individual-emergent themes specific to the initial stages of advanced mathematical-development. This was repeated for Diary Phase Two, resulting in a second set of individual emergent-themes specific to the later stage. Any data from the interviews which had not already been identified as clarification of earlier lived experience were then analysed to uncover the individual's phenomenological understanding of their evolution as an individual gifted mathematician engaged in advanced mathematical-development. This resulted in a third set of individual emergent-themes pertinent to their reflections on advanced mathematical-development.

Despite adopting three phenomenological lenses for each participant, my professional interpretations of each individual's data as practitioner and researcher were minimised at this stage, in line with the epistemological principles which hold the individual as the person best placed to describe their experiences in the first instance (3.4). The extent to which it is possible for a researcher to completely shield a phenomenological analysis from their own assumptions and perspectives is the subject of fierce debate (Ayton, 2023). It is generally considered more difficult to achieve this when the researcher is investigating an issue close to their own practice, which they are more likely to have stronger preconceptions of (BERA, 2018). This is particularly applicable given my personal and professional experiences as a gifted mathematician (1.5). However, I observed several practices in the study to protect the phenomenological spirit of the analysis. To facilitate member checking (3.10.2) some of the data were analysed before all the data were collected. This was a departure from the initial phenomenological approach within a typical interpretative-phenomenological analysis, in which all of each individual's data would be analysed collectively only after it was all generated (Smith, Flowers & Larkin, 2009). In the study, the data were analysed separately for each participant. However, within each participant's phenomenological analysis, data from each of the first two data-generation phases were further separated out as described above, and initially analysed distinctly. Moreover, several distinct phenomenological lenses were adopted within each phenomenological analysis. This was to understand the individual's evolution as a gifted mathematician immersed in advanced mathematical-development, particularly as it related to them overcoming the anticipated early challenges within that experience (2.5.3, 2.5.4). Finally, the concluding interviews enabled each participant to comment on the way their diary data were being interpreted, and hence to guide the phenomenological lenses being applied to their data. Participants were told my initial interpretations of their diary data during their interview. If they felt I had not captured their perspective faithfully, they therefore had a specific opportunity to re-polarise the analytical lens by clarifying how their intended meaning differed from the interpretation being put forward. Alternatively, they could simply confirm the interpretation aligned with their perspective if they felt it had been faithfully captured.

Once the three chrono-phenomenological analyses had concluded, the analytical approach proceeded in much the same way as a traditional interpretative phenomenological analysis (Smith, 2017). This was the interpretative stage in the analysis where all participants' data were collectively analysed and interpreted to form shared themes, which apply more widely (Grandy, 2018; Hyde, 2020; Ibarra & Adorjan, 2018; Löhr, 2021; UoA, 2024). By this stage, I had become the person with the most-complete knowledge of the views each individual had contributed throughout the research; not only the participants', but also my own. My objective was to utilise this knowledge throughout the subsequent interpretative analysis, and in doing so ensure every individual's perspective was fairly represented when creating the shared themes. This involved harnessing the power of my own perspectives as a gifted mathematician and teacher of other gifted mathematicians (1.5, 3.3). I consider that my experiences endow my perspectives with a high value in this regard in that I have experienced advanced mathematical-development both as a gifted mathematician and a teacher (3.4). However, it was anticipated the participants would interpret some aspects of their experiences very differently than I would, making it especially important to adopt a reflexive (3.3) approach to analysing and writing about the findings (3.14).

3.13 Pseudonym Creation: Encapsulating a Gifted Mathematician's Phenomenological Spirit

The participants were initially referred to as Participant One, Two, and Three within the data prior to analysis, as anonymisation was necessary in line with the approved ethical procedures (3.7, Appendix Three). However, these labels were impersonal, failing to capture the phenomenological spirit of the person they represented. One method which enables a researcher to continue thinking of their participants as distinct individuals is to use their real name throughout the analysis, only pseudonymising after the findings and analysis have been written entirely (Wang *et al.*, 2024). However, this was problematic given my status as practitioner-researcher (3.7). As my participants' teacher, I knew them as people independently of my study in a way a researcher generally would not. This meant I could easily conflate what they had told me about their advanced mathematical-development throughout the research specifically with what I knew about them as people more generally. Pseudonymisation therefore aided my observation of this distinction within the interpretative analysis by distancing myself from the people they were more broadly, that their real names represented.

Consideration was given to asking participants to choose their own pseudonyms (Itzik & Walsh, 2023). Allen and Wiles (2016) suggested this approach, further concluding that the process of naming is not only of personal value, but an important act of research in its own right. Prior to the interpretative analysis, I had already developed a detailed understanding of who each participant had grown to be throughout their advanced mathematical-development. These understandings arose naturally through the chrono-phenomenological analyses (3.12). Moreover, these understandings were based solely on the research data. It was this aspect of the participant's experiences that I sought to respect within the subsequent interpretative analysis. I therefore decided to select a pseudonym for each participant whose meaning encapsulated my understanding of who they became through their advanced mathematical-development (Lahman, Thomas & Teman, 2023). The description of each chrono-phenomenological analysis therefore concludes with a description of how the pseudonyms were selected (4.2.5, 4.3.5, 4.4.5). Pseudonym selection was inherently subjective. Justifying the choices for each participant therefore signified the introduction of my interpretation at the correct point in the data analysis (3.12), when chrono-phenomenological findings were to be interpreted collectively to facilitate wider understanding. This ultimately resulted in the pseudonyms Confur, Derwyn, and Ethan for the participants, and Kindred, a second teacher mentioned by Ethan.

3.14 Writing Strategy: Presenting and Critically Evaluating Perceptions of Advanced Mathematical-Development

An independent findings chapter associated with an interpretative phenomenological analysis would typically present both the individual emergent and shared themes through including relevant excerpts from each participant's data alongside explicit mention of the researcher's interpretation of what the participants meant when generating said data (Smith & Osborn, 2007). However, this was not optimal in my study for two reasons. Firstly, my existing relationship with the participants (3.8) meant that extra care was required to avoid conflating what they told me about their advanced mathematical-development throughout the study with what I knew about them as people more widely (3.7). It was therefore advantageous to undertake each chrono-phenomenological analysis, and to represent the associated findings, in the transcendental spirit which places greatest epistemological value on the individual's own interpretation of their experiences (Cheng, 2024) in the first instance. Purposefully presenting each gifted mathematician's individuality through writing about their findings from their own perspectives meant that I remained conscious only to represent what they had told me about advanced mathematical-development throughout the study, and hence avoid the aforementioned conflation. Secondly, one purpose of the developed chrono-phenomenological analysis procedure was to evaluate the gifted mathematicians' *lived* experiences of both the initial stage of advanced mathematical-development and after a period of adjustment. When writing up the chrono-phenomenological analyses, it was therefore essential to represent each participant's lived experience of advanced mathematical-development during both diary phases, and to clearly differentiate their lived experience from their subsequent perceptions of earlier experiences in hindsight. To faithfully represent each participant's worldview in the thesis, the sections pertaining to each chrono-phenomenological analysis therefore had to demonstrate how data pertaining to all three data generation phases (3.6) were interpreted through three distinct phenomenological lenses, emphasising their lived experienced during both diary phases separately, and their subsequent hindsight reflections from their interview.

The individual emergent-themes associated with each phase of data collection are, therefore, first detailed and justified independently for each participant in the Findings chapter (4). To aid clarity when presenting these subtly-different elements, there is no attempt to compare or contrast the participants' perceptions, offer my interpretation of their data, or critically evaluate the findings in light of the literature at this stage. Instead, each participant's own interpretations are emphasised as far as possible, with distinctions drawn only between the lived experience and hindsight reflections of each individual.

The interpretative analysis, which considered all participants' data collectively (3.12), was a completely separate phase in the analytical procedure. It is therefore presented distinctly to the three chrono-phenomenological analyses, after they are detailed (4). The Discussion and Analysis chapter (5) presents the shared themes which were crafted during the interpretative analysis. The findings are critically evaluated by detailing how perspectives from the literature, and the application of the theoretical framework (2.6), were utilised to inform the collective interpretation (Wilmot, 2023).

The Discussion and Analysis chapter goes further than simply expositing the interpretative analysis and critically evaluating the findings in light of the identified literature (2). My perspective is also actively included. This undoubtedly represents a divergence from both the traditional approaches to presenting analysis in doctoral theses (Nayak *et al.*, 2023), and the objectivity within writing I am naturally predisposed towards as a gifted mathematician (3.3). Doctoral researchers have historically sought to remove their voice altogether in analysis chapters (Paltridge & Starfield, 2020), in an attempt to shield said analysis from the corruption they perceive in their own subjectivity (Borraz, Zeitoun & Dion, 2020). Many scholars have therefore taken the view that researchers should not arrogate to themselves the privilege of including their own perspective in their academic writing, lest they detract from their work's rigour (Adler, 2022). However, Weatherall (2019) argued against the automatic adoption of the traditional doctoral-thesis structure and the typical styles associated with its usual chapters, advocating instead for formats and styles that best support the presentation of the nature and nuances of the study it is written to represent. She (*ibid.*) put forward that utilising the traditional structure without careful consideration of its appropriateness for representing a specific research project can become obstructive. In particular, she (*ibid.*) held that traditional styles of doctoral-thesis chapters can be especially constraining when the study involves a degree of researcher subjectivity. In this case, passive voice can make the researcher's subjective interpretations appear as though they were fact or objective conclusions, making active voice clearer and more accountable (*ibid.*). Researcher subjectivity has traditionally been a particular concern in that it undermines a study's generalisability to other cases (Varpio *et al.*, 2021), and hence has been routinely avoided in doctoral writing. However, my study, like all interpretative phenomenological analyses, is idiographic (Sekar & Bhuvaneswari, 2024). Like other qualitative studies, my research was not therefore overly preoccupied with providing a fully-generalisable understanding of the experience of advanced mathematical-development (Evans, Carlyle & Paz, 2023). Instead, its focus is on learning as much as possible from the small number of carefully-selected gifted mathematicians who participated (3.8). Eliciting their detailed perceptions of advanced mathematical-development (Buhagiar & Sammut, 2023) gave rise to potential considerations practitioners might weigh when selecting and employing pedagogical practices (Orinov *et al.*, 2021) to support advanced mathematical-development effectively. In this way, definite answers about which choices are objectively the best in specific scenarios are not necessary for the creation of a pedagogical model. The inherent subjectivities of the participants and me as gifted mathematicians are therefore celebrated within the findings and analysis, rather than excised (Ajjawi *et al.*, 2024), as motivators of the pedagogical model arising from this research (6.3).

Author self-mentions are essential in writing pertaining to interpretative phenomenological-analyses as clear demarcations between the participant's intended meaning and the researcher's subsequent interpretation of it (Delve & Limpaecher, 2023). However, this demarcation was further supported in the thesis by positioning the outcomes of the chrono-phenomenological analyses within their own chapter (4), separating them from the subsequent interpretative analysis. Hence, researcher self-mentions are purposefully limited in the Findings chapter, occurring only where a participant referenced me directly. However, author self-mentions serve another purpose in constructing and communicating a researcher's identity to evaluate its implications for a study's subsequent recommendations (Hardjanto, 2022). Wang and Hu's (2023) perspective that researchers obscure their work's subjective nature through limiting author self-mentions is therefore a criticism of an analytical style my writing sought to circumvent. My own interpretations and preconceptions were therefore explicitly considered throughout the Discussion and Analysis chapter. Acknowledging my relationships with participants (Lehman & Tienari, 2024) and personal experiences of advanced mathematical-development in their own right (Hibbert *et al.*, 2014) enabled my assumptions (3.4) to be regularly stated and challenged, supporting the rigorous formation of recommendations for wider pedagogical practice (van Beveren, 2024). My views on these issues are so prevalent (Engward & Goldspink, 2020) they required ongoing acknowledgement and explicit challenge to ensure they did not overpower the participants' perspectives (Pihkala & Karasti, 2024).

I am a gifted mathematician with my own experiences of advanced mathematical-development (1.5). Therefore, my subjectivity is, of itself, another valuable answer to research question one, which asks how gifted mathematicians perceive advanced mathematical-development (3.2). My subjectivity as a teacher of gifted mathematicians in FE settings also helped address research question two, which asked how pedagogical practice could be developed in light of gifted mathematicians' perceptions. The choice to include researcher perspective within the critical analysis facilitated further answers to the research questions, reflecting Weatherall's (2019) argument that writing styles must be adapted to support the faithful exposition of a specific study. However, the extent to which a researcher should write themselves into their research requires careful consideration (Robson, 2024; Thomson & Kamler, 2016); in particular, to ensure the practice serves an academic purpose through aiding the analysis rather than hindering it. My interpretation does not, therefore, dominate the analysis of gifted mathematicians' data; neither is the analysis entirely researcher-centric. Rather, my perspective remains present within the analysis by explicitly detailing reflexivity considerations throughout the interpretation of the data and presentation of the shared themes (Sternad & Power, 2023). Simultaneously, the established literature base is applied to elevate this analysis to a critical evaluation (Crossley, 2021), in particular through detailing how existing scholarly perspectives and the theoretical framework informed the way I interpreted the findings.

3.15 Summary

This chapter has established the approach to harvesting the perceptions of gifted mathematicians during their further education and how to subsequently utilise these perceptions to better understand advanced mathematical-development. The Findings chapter presents these perceptions by chronicling each participant's narrative in detail.

4 Findings: Three Gifted Mathematicians’ Perceptions of Advanced Mathematical-Development

4.1 Introduction

The intention of the Findings chapter is to present the outcomes of the three chrono-phenomenological analyses which were undertaken for each participant. In doing so, the chapter exposits three detailed perceptions of advanced mathematical-development. This answers the first research question, which asked:

1. How do gifted mathematicians perceive their experiences of advanced mathematical-development throughout the further education phase?

These perceptions are intentionally presented from each participant’s phenomenological perspective, with no attempt to compare or contrast them. Hence, this chapter predominantly addresses the first research question, with answers to the second preserved for the Discussion and Analysis chapter (5).

In the spirit of the chrono-phenomenological analysis approach (3.11) adopted to first consider the participants’ experiences of advanced mathematical-development from their individual worldviews, the Findings chapter is offered in three sections. Taking each participant in turn, a brief educational background is given to contextualise the outcomes of the chrono-phenomenological analysis pertinent to them that follows. Specifically, the individual emergent-themes are presented with respect to the three data-generation phases separately. The chrono-phenomenological analysis associated with each participant is then presented, starting with individual emergent-themes pertaining to Diary Phase One (January 2023), to establish the participant’s lived perceptions of their “experience *embarking* on advanced mathematical-development”. Next, the analysis of data relating to Diary Phase Two (June 2023) is likewise exposited to establish the participant’s lived perceptions of their “*maturing* experience of advanced mathematical-development”. Both of these sections reflect subsequent discussions with the participant that took place formally during their interviews (November 2023 – January 2024). The way I had understood their intended interpretation of their earlier data was presented to each participant. This allowed them to comment on the accuracy of the interpretation, and hence served to either confirm the phenomenological lenses had been accurately applied or to re-polarise them, and in doing so validate the findings (3.10.2). The findings presented, pertaining to experiences which happened during Diary Phases One and Two, therefore also account for any relevant contextualising data offered during later stages. In particular, data generated at the Interview Phase is sometimes presented where it clarifies their lived experience at earlier times. Finally, hindsight reflections are presented through any new individual emergent-themes from the analysis of the participant's Interview Phase data (4.2.4, 4.3.4, 4.4.4), detailed in a subsection called "*reflections* on advanced mathematical-development". To conclude the presentation of each chrono-phenomenological analysis, the perceptions of advanced mathematical-development that each participant's presentation of findings represents is summarised. The choice of pseudonyms for each participant is then explained accordingly (4.2.5, 4.3.5, 4.4.5).

All participants provided rich data in their diaries and interviews. However, the volume of data at each phase varied between the participants. The sections relating to the various stages of the participants' chrono-phenomenological analyses therefore somewhat differ in length. There are also striking similarities between the participants' individual emergent-themes discussed in each subsection. The in vivo approach (3.11) to data analysis meant individual emergent-themes ultimately arose in each participant's own words. Referring to similar ideas with different vocabulary is therefore deliberate within this phenomenological presentation of findings.

The participants referenced five people: themselves, referred to within the data via the first person, and by their pseudonyms, Confur, Derwyn, and Ethan; me, referred to by the participants through the second person or by my first name, Niall; and a physics teacher they frequently interacted with, pseudonymised as Kindred. Each individual's references to these people are purposefully presented objectively in the chrono-phenomenological analyses by restricting any such mention to the exact words used by the participant. Once the findings for each participant have been detailed in full, the Discussion and Analysis chapter (5) presents the cross comparison that resulted from the interpretative analysis (3.12). At this point, the three perspectives on similar experiences will be conjoined by developing all the individual emergent-themes into shared themes, which are relevant to all participants.

4.2 Confur’s Experience of Advanced Mathematical-Development

4.2.1 Confur’s Educational Background

The first participant is referred to as Confur. He was 16 at the beginning of the study and is white-Indian, having lived in the UK since birth. Confur attended an 11-16 state comprehensive, joining the mathematics school after an anticipated transition to a standalone sixth form college at 16. His aspiration was to study mathematics at university; he applied to the University of Oxford.

4.2.2 Confur’s Experience Embarking on Advanced Mathematical-Development

When the problem-solving sessions commenced, Confur immediately focused on the new style of problems he was encountering. The first theme that emerged therefore pertained to his adjustment to the new abstract style of questions. The views he expressed during Diary Phase One were often contextualised by his perceived successes or failures with solving them. He noted specifically that such questions were ‘*more abstract but more rewarding’*, and stated he ‘*enjoyed the difference between the A-Level questions we normally do to this’*. Confur further clarified this when interviewed:

> *I definitely enjoyed [the problems] [be]cause [they were] interesting. But I also think the challenge is enjoyable. (…) I had obviously never done anything like it [before] and there was a lot of maths that I didn't quite understand. I think that (…) made me want to learn more about it, because it (…) [did not] make sense. But then, to [some] extent in my mind, it's like, oh, there's not much about this I know. I want to find out [more].*

In the earliest stages of his advanced mathematical-development, Confur therefore felt intrigued by the difficulty of the problems. Encountering problems actually vicinal to rather than beneath his ZPD piqued his interest. However, Confur had a variable view of how he perceived the experience of tackling the problems during the early stage. He subsequently recorded feelings of frustration related to not knowing how to begin appropriately-challenging problems. He wrote that '*I seemed to always need the first step given to confidently know what to do'*. Confur made the following similar comments towards the end of Diary Phase One: *'I was stuck with where to go in questions and seeing the next step. I always seemed to need the first step given to get going'. This is always tough but happens often.* Confur further detailed that feeling stuck in this way often led to strong emotions which affected his perceptions of advanced mathematical-development profoundly, both in the moment and in a more-longstanding way. After one session he wrote '*I misunderstood an explanation (…). This was stressful.'*, detailing that he found the session *'demoralising'*. Following this session, he further wrote that *'I am nervous about the next session'* and stated that he *'wasn't very motivated to continue onto other questions without help'*. Nevertheless, Confur went on to reflect on his experience overcoming this feeling of frustration with one of the problems. He stated that he '*really enjoyed the style of question'* when he made progress, stating that *'this was nice and really helped me understand more easily'*. This led to Confur also expressing a more-hopeful outlook. He wrote that, despite previous feelings of stress and demoralisation, he thought he would become *'much better when [he got] used to the style of questions'*. He was subsequently asked about what he thought helped him to start making this progress at his interview. He stated that:

> *I think you have a very good way of explaining things when I [did not] understand them because I think you can explain them in a lot of different ways. (…) But I also think sometimes you just [have] to cycle through until something clicks. You've got to keep banging your head against a brick wall. And then all of a sudden it gives way. Oh, and then your head doesn't hurt anymore.*

In the early stages, Confur therefore benefitted from two contrasting approaches to overcoming challenges with the problems. A direct explanation from me immediately plugged a gap in knowledge or skills. Working independently had a more-profound impact but took sufficient determination on Confur's part to persevere when struggling. In relation to how he overcome the challenges of working independently, Confur also stated that *'I think a key part of it was (…) looking at the solutions and trying to understand them rather than just looking at the answer.'*. This is another key distinction between how Confur assessed himself during advanced problem-solving as opposed to when solving questions at an A-Level standard. At that level, he considered the final answer to be the most important thing. But, for higher-level problems, it was the solutions, in other words understanding the method, that had become Confur's focus.

Confur reflected on how he felt when stuck in the context of how quickly he understood how to solve a given problem, becoming the second emergent theme. He wrote about frustrations when he judged his understanding was slow. Likewise, Confur's perceptions were more positive when he believed he had *'caught on quickly'*. The essence of how Confur's perceptions of his own speed influenced his perceptions of advanced mathematical-development is captured in his statement that *'I am proud of my understanding and the speed at which I got it. I didn't feel like I particularly slowed down at any point.'*. This quote suggests Confur's perceptions of advanced mathematical-development had been impacted positively by the experience he was writing about. However, it also demonstrates that he used his perception of his speed of understanding as one criterion against which he judged his success during advanced mathematical-development. He likewise reflected on his speed of understanding acting as a negative influence, describing one session as *'demoralising and stressful because of the speed'*. Confur made comments of this nature commonly, also stating that: '*I fell behind in the last two sessions and have consistently struggled on understanding. (…) The problems are always very interesting when (…) I understand but otherwise are very tough.'*. There were, however, also occasions where Confur described experiencing slower understanding as beneficial. For example, he wrote that *'the question (…) was very confusing so I went away and watched someone else explain slowly and made sure that I mostly understood the question. It was interesting when I got it'*. Moreover, Confur indicated an experience where he turned his initial frustrations into success by perceiving his speed of understanding in a more-nuanced way. He wrote that *‘At times the speed of explanations confused me but when I read through slowly it helped’*. When asked about what he believed influenced his speed of understanding during his interview, Confur stated that when he *‘was doing it consistently [he] was maintaining [his speed]. The times [he] he did less were the times [he] slowed down’*. He therefore identified that regularly practising his problem-solving helped him sustain a high speed of understanding that in turn created positive perceptions of advanced mathematical-development.

The third theme which emerged related to Confur feeling he stood out from his peers. Confur made explicit reference to his perceptions of his own ability changing when he judged himself relative to peers. When he perceived something which set him apart positively from his peers, he felt the effect keenly. For example, he wrote: ‘*I was proud of how I understood parts of the question in a different way than (…) it was explained. I thought I looked at it more visually than everybody else, which I was proud of.’*. Confur was referring to a question about vector representations of lines. Questions about vectors often share an uncommon feature, that they can be tackled geometrically as well as algebraically. When Confur stated he *‘looked at it more visually’*, he was indicating that he was the only one to tackle the problem geometrically; the other participants and I pursued an algebraic approach. Confur’s solution was not only more sophisticated mathematically, but he also arrived at it without any assistance. This is an example of a beneficial perception. However, he also experienced this in a negative way. For instance, he wrote that *‘The problems are (…) very tough. (…) I also asked some friends for help, but it didn’t change much'*, further explaining afterwards that he felt less *‘motivated to continue with the questions without [further] help’* as a consequence. When asked about how other participants influenced him in the early stages of the study at his interview, he went on to talk about this further. He stated that working alongside peers at this time was often:

> *negative, because sometimes I believed other people were finishing questions first or faster. And I think sometimes if everyone's doing the same question*

> *and there's people around me talking about how they got to the answer, and I'm not even halfway, that (…) wouldn't motivate me. (…) it was a bit of a negative, [but] it was never like detrimental.*

Confur found it challenging when he believed other gifted mathematicians were faster than he was at working through the problems. However, this experience was not so negative that it completely demotivated him when he looked back upon it retrospectively.

4.2.3 Confur's Maturing Experience of Advanced Mathematical-Development

Confur's perceptions of his speed of understanding were also evident within the perspectives he shared during Diary Phase Two, becoming an emerging theme relating to his growing sense of independence. At the end of the first session, he reflected that he '*felt very confident with [his] answers and knew where to go at every step with a bit of thought. It was very satisfying to come to the answer first try.'*. Previous interactions with Confur had revealed that problems had typically taken longer because it took him several attempts to arrive at an appropriate method for that question. I therefore interpret his reference to *'answering on the first try'* to be his perception that he was now faster than he was during Diary Phase One at discerning a valid approach. However, he had appeared to have grown beyond using speed of understanding as the predominant benchmark by which he judged himself. His comments about the questions themselves and his perception of his ability to tackle them had evolved. The focus was now much less on adjusting to the challenge; this adjustment had taken place between Diary Phase One and Two. His diary entries at this point pertained more to his confidence to try the problems more independently or with less help. For example, Confur wrote that *'I was proud when I answered the question correctly and it felt easier, like something had clicked in my method of approach'*. This was just one indication that the way Confur was perceiving advanced mathematical-development had changed. He no longer reflected negatively on the challenge inherent in the advanced mathematical-problems he was working on. Instead, he had begun to perceive this challenge as beneficial.

By Diary Phase Two, I had adapted the approach to which questions were selected for the problem-solving sessions, and the participants had some autonomy to choose for themselves. The evolving approach to problem selection is further detailed in 3.9.1. Notably, in some of the sessions which took place during Diary Phase Two, Confur made a conscious choice to select a different question than his peers. This happened even when the rest of the group chose to collaborate on the same question. In a corresponding diary entry, Confur made the following comment: *'I felt fairly challenged but overcame my challenges confidently and quickly. Everyone else was on a different question so I worked independently, and it went well'*. Confur's perception of how independence affected his perceptions of advanced mathematical-development had therefore also developed a new nuance since Diary Phase One. He was, perhaps subconsciously, not only to some extent more aware of how peers influenced this, but gravitating towards the positive influences and away from the negative ones. Originally, Confur's perceptions were impacted profoundly by the presence of peers (4.2.2). When he believed he stood out in some obvious way, this either improved his perception when he believed he was singularly successful, or hindered it when he judged himself poorly. These judgements were made in the context of all participants collaborating on the same problem, where Confur could gauge his ability based on his relative speed of understanding and how the participants arrived at answers with varying success and via various methods. Confur could form a judgement on being quicker, smarter, or more successful than peers with more ease. This suggested his perceptions of advanced mathematical-development fluctuated readily at this earlier stage. By purposefully choosing to work on different questions than his peers, Confur created new opportunities to demonstrate to himself his ability to be successful independently, whilst limiting his ability to compare himself negatively. Whether other people were successful more often or more quickly than Confur became almost irrelevant to him given that they were tackling a different problem. This is not to say that Confur's perceptions of advanced mathematical-development stopped being influenced by the presence of peers. Rather, by choosing to distance himself in this way he was mediating the impact of the rest of the group to maximise its beneficial influence. This is explored in more detail in the later analysis (5.4.5).

4.2.4 Confur's Reflections on Advanced Mathematical-Development

Most of what Confur shared in his interview served to clarify the views he expressed during Diary Phases One and Two, and to validate my interpretation of them. However, there was one individual emergent-theme that did not appear evident in his diary entries when analysed initially. When asked what aspects of his experience of advanced problem-solving he thought had been most useful for his progress, Confur stated that *'times I was [solving problems] like set in stone (…) especially coming up to Mathematics Admissions Test [MAT], because I was doing loads of problems everyday up to then. When doing it consistently I was maintaining it.'*. Confur had worked on his preparation for the MAT consistently since the outset of Year 13 in September. Preparation for subsequent interviews continues beyond this time, usually until early December. Confur is therefore describing a period of three months in the year where solving advanced mathematical-problems was a sustained part of his daily life. He said that *'When [he] was getting ready for the MAT [Mathematics Admission Test], [he] remember[ed] being very excited and motivated the first time [he] did (…) a full paper in the [allotted] time, marked it and was [over] the threshold'*, noting this as an indicator of significant progress. When asked what he would want to change about his experience of advanced problem-solving if he could go back and do something differently, he stated that:

> *I think I would have started doing more earlier because (…) it was relatively late in when I realised nothing was going to happen if I wasn't as consistent as (…) I was [in] the end. But other than that, I don't think there's much else. I don't know, probably just doing more questions and varying the types of questions.*

4.2.5 Summary of Confur's Advanced Mathematical-Development

When Confur began working on appropriately-challenging problems, he was a curious mathematician who found it enjoyable to feel genuinely challenged. In particular, he noted a difference in style compared to A-Level problems. He quickly had experiences of struggle with the questions, and his notions of understanding things at speed influenced his perceptions of his own success with advanced mathematical-development. He learned over time that the problems did not require quick solutions to be enjoyable or rewarding. He therefore gradually became more reasoned in his approach to them. Confur experienced a challenge associated with sharing the experience of advanced mathematical-development with other participants. In particular, he compared his own performance with peers who were working on the same problem. He realised that he sometimes doubted his own ability if he thought the other participants were slightly ahead of him. Likewise, he benefitted greatly when he thought he had been faster or more ingenious than his peers. At some point Confur noticed this and began turning it to his advantage. He actively chose different problems to the other participants, to inhibit his ability to directly compare his progress with peers. In this way, he to some extent let go of how quickly he was succeeding. Moreover, he no longer attributed his own successes to the presence of other gifted mathematicians. He created a space where he could celebrate his own achievements and hence sustain positive perceptions of his advanced mathematical-development in his own mind without needing anybody else to acknowledge it directly. I hence chose the pseudonym Confur, meaning humble yet talented, to encapsulate this aspect of his being that was so prominent to me in the perspectives he provided throughout his advanced mathematical-development.

4.3 Derwyn's Experience of Advanced Mathematical-Development

4.3.1 Derwyn's Educational Background

The second participant is referred to as Derwyn. He was 17 at the beginning of the study and is white-British, having lived in the UK since birth. Derwyn attended an 11-18 state grammar school, joining the mathematics school after leaving his previous school earlier than anticipated at 16. His aspiration was to study natural sciences or physics at university; he applied to the University of Cambridge to read natural sciences.

4.3.2 Derwyn's Experience of Embarking on Advanced Mathematical-Development

Derwyn made regular reference to enjoying tackling the problems during Diary Phase One, becoming the first emergent theme. The following quote, recorded immediately following the first session, captures his nuanced perception of this experience:

> *I enjoyed tackling questions that weren't part of [the] regular curriculum, [and] seeing questions that asked about situations (…) not in an A-Level way (…). STEP (…) feels like (…) maths with no crutch (…), so I really enjoyed the challenge. (…) If I was to see a question in this niche (…) I'd have an easier time structuring my answer, so I feel like I've grown.*

Derwyn later went on to state that *'being able to answer such questions with little or no practice really [made him] feel like [his] mathematical abilities [were] improving.'*. He also clarified what he perceived as the difference between A-Level problems and advanced problems. He said he believed that in *'A-Level questions, you'd get told [the method], and you learn it. You sort of just follow up, using the "whole brand" scheme of maths.'*. The words *'whole brand'* when applied by Derwyn to mathematics problems are in reference to the way he thought A-Level questions were designed. They tested specific topics in specific ways, rather than mastery of lots of topics in novel ways. For this reason, Derwyn did not always immediately know how to solve all of the problems. Entries made during Diary Phase One relate to times where he had to think them through carefully. For example, he wrote of another question that *'the aspect [he] was most proud of was not being discouraged by such a hard question. (…) [he had] observed the enjoyability of tackling such tough questions, [which] motivated [him] to tackle similar ones independently.'*.

Right from the first session, Derwyn worked eagerly with peers, Ethan in particular, and the way he used them to build confidence became the second emergent theme.

In that session, we tackled a problem on polynomials with unknown coefficients. It required the unknowns to be inserted in factorised form, rather than the more-typical expanded form encountered in A-Level questions. Derwyn did not find this immediately straightforward to notice. However, he did manage to solve the first part with help. Reflecting upon this, he wrote the following: *'I did feel a bit stuck. (…) I think working with Ethan made the environment very easy and since I believed we could be able to do the question since two people makes the problem easier, it in turn made me less nervous'*. Derwyn therefore found his peers useful within the problem solving, not just for making progress with the problem itself, but also for inspiring confidence in his own abilities. In fact, Derwyn identified specific ways collaboration helped him transcend his ZPD. He reflected that *'I was stuck because I looked at the problem and focused solely on the answer I wanted to get not patterns that were already there.'*. Ethan subsequently spotted one of these patterns. Derwyn talked about the impact of this help, continuing by saying that *'With help though, this idea really helped me in tackling the second part of the question later on.'*. He therefore reflected on collaborative work that ultimately influenced his independent success at later times.

4.3.3 Derwyn's Maturing Experience of Advanced Mathematical-Development

Derwyn continued to write often about the questions themselves during Diary Phase Two. His enjoyment of the problems was still evident in this phase, hence becoming the first emergent theme. However, his preoccupation with getting to grips with methods and approaches for tackling the problems in the sessions had moved on. Instead, he now focused less on how tackling the problems impacted his perceptions of his progress, and more on how it was influenced by regularly encountering these problems in many guises, and fully solving them. He encapsulated the effect of these combined influences when he wrote that *'Since seeing how fulfilling completing a whole question is, I am quite motivated to complete similar questions with similar rewards.'*. This is explored further below in relation to the two categories of codes through which this understanding arose: "Stepping Back from the Problem", and "Readiness for New Challenging Questions".

Derwyn often described tackling the problems enthusiastically in earlier diary entries. However, during Diary Phase Two he wrote of a more-reserved mindset. Earlier references to heading quickly into the problems were no longer prominent, having been replaced with a deliberate slowness. A slower pace created Derwyn's thinking space to problem-solve more effectively. Talking about a problem he encountered outside of a session, he stated that *'It felt quite slow at first, but after I had taken a break and come back to it, it felt much more doable.'*, and went on to further explain that *'assessing what to do (…) made me feel very engaged and happy with the task.'*. The way Derwyn perceived and used help to develop confidence in mathematical problem-solving had undergone an analogous evolution. He no longer valued support on the basis that *'two people make the problem easier',* improving their collective chance of success. Instead, Derwyn valued help which enabled him to step back, think about the problem, and then solve it independently. Derwyn stated that *'help was good when it made me step back and think of the problem in another way (…) which led to me getting the correct answer.'*. He further expounded on this in the following diary entry: *'Seeing [Kindred] take a second to think before answering the question (…) makes me feel like I don't need to do everything with 100% efficiency, just being able to take on problems with an open mindset is the most important thing.'*. Derwyn would go on to talk about this at length at his subsequent interview. When asked about his experiences of stepping back to create the mental space needed to solve a problem, he said the following:

> *I don't know when the mindset I'm in is the right one. When I'm going away and changing the mindset and going back with a different mindset, it's hard to tell whether the first attitude was right or wrong. There's no clear line. (…) you could work for ages (…) [thinking] that [this] is the right mindset. (…) But through the nit and grit, you might think, oh, this is taking too long. This is obviously the wrong mindset. Go back with different mindset, as it [might] now be possible. (…) when you come back to it, I think a lot of the time [that] is good, chances are, if you're finding it difficult it could be easier. (…) when you come back with a different mindset, it could help. But there [have] been time[s] when I've done a question, and it's been like, oh, this is hard (…) and I come back with a different mindset (…) and then the second time still got it wrong, and (…) now I've got it wrong twice.*

Derwyn's view of this in hindsight, then, was not as consistently positive as it had appeared to be based solely on his perspectives from Diary Phase Two. It only seemed as though resetting his mindset was particularly helpful because that was the nature of the experiences that happened during Diary Phase Two. In truth, he did sometimes find resetting his mindset unhelpful, particularly if it did not work first time. However, whether doing this helped or hindered Derwyn, he was able to identify when he had spent sufficient time on the problem without making the progress he hoped for. In this way, Derwyn sustained positive perceptions of his advanced mathematical-development. He knew that his best chance of success was to stop when his efforts had become unproductive, as his mindset was likely to be wrong. His knowledge that there was a small chance this might backfire further helped him, as he was consequently less perturbed by this when it did happen.

Derwyn also reflected specifically on how perceiving a development in his approach to problem-solving affected his perceptions of advanced mathematical-development: '*I am most proud of seeing questions I had seen previously (being unable to complete) and understood it fully and being able to do it. (…) Seeing this change really makes me feel like I am improving my mathematical ability, so I'd like to improve further.'*. Derwyn consistently sought out these opportunities to further improve throughout Diary Phase Two, both during and separately from the problem-solving sessions, even whilst acknowledging that his skills had developed significantly. He stated that *'I am quite practised in harder questions (…) I still feel like I can improve quite a lot. Coming up to a session I feel very excited/ready for new challenges.'*. Derwyn therefore transcended his ZPD incrementally by solving ever-harder questions. He was no longer perturbed by problems that seemed difficult. He reflected on this further, stating *'I saw just how difficult even harder questions can be, which motivates me to get better, in order to tackle these questions in the future.'*.

The second theme emerged when Derwyn described two predominant feelings throughout this process of honing his mathematical problem-solving skills: frustration and excitement. Moreover, Derwyn detailed his perception of a nuanced relationship between them. He did not automatically perceive feelings of frustration in a negative way. Rather, Derwyn sensed opportunities to improve when he felt frustrated. Hence, he often perceived excitement within frustration. Derwyn wrote that *'I felt frustrated when my knowledge came to an end and I could no longer complete the question. (…) However, it makes me feel very excited for further maths where I can progress and eventually (hopefully) be able to tackle any of the questions.'*. On occasions, Derwyn was even able to expedite his transition from frustrated to excited though sustained concentration. He stated that *'I felt slightly stuck when I hadn't concentrated. (…) I tried to fully concentrate even when I thought I knew it already, so I never got stuck.'.* Derwyn therefore recognised complacency as one factor leading to moments of frustration. By regulating his thoughts and feelings in these moments, he positioned himself to think more positively about his ability level, improving his perception of his own advanced mathematical-development.

4.3.4 Derwyn's Reflections on Advanced Mathematical-Development

The first theme which emerged related to Derwyn's pursuit of personal goals. When Derwyn was asked which problems he was proudest of solving during his interview, he talked at length about how he approached his admissions interview at the University of Cambridge. He said that:

> *I was quite proud of [what] I discussed at my interview. (…) I got some, you know, out of nowhere questions (…) and (…) it felt like I had basically been thrown in[to] something that I had literally no idea [about]. (…) It was empowering in a way, because (…) I was able to create something and derive something that I hadn't been told before. A lot of A-Levels questions, you'd get told [the method], and you learn it. You sort of just follow up. (…) But [at the interview] I felt like I actually made something, and I did something. You know, it felt like I wasn't just told it. (…) I was quite proud of that.*

Derwyn had been working on mastering the skill of solving the advanced mathematical-problems, of which many are described as novel or unusual (Siklos, 2019), since the onset of the study. His belief that he was able to demonstrate this mastery to a respected academic in an interview scenario was therefore a significant achievement to Derwyn. He also went on to specify another achievement associated with his university application that had made him feel very proud. Derwyn spoke of preparing for the Natural Sciences Admissions Assessment[2] (NSAA), an exam undertaken at the end of October. The outcome is considered when the university decide which applicants will be invited to interviews. Derwyn said that:

> *There was also the volume of work I did for the NSAA. I was quite proud of that. I look back at all the work I did, and I was like, I haven't let myself down. (…) I've put in work and felt quite good. (…) Me and Ethan decided that we would need to get some work done. So, we spent a lot of our frees doing that. And after school on the weekends we went to the library.*

Derwyn was therefore proud of the consistency with which he worked on all aspects of his application, the time he had dedicated to it, and the skills he had developed as an advanced problem-solver along the way. His goals during this period were some of his important motivators. He subsequently reflected on the progress he made throughout his preparation for the NSAA, making the following statement which demonstrates how he found motivation in finding it challenging initially:

> *When I first looked at NSAA I started by just taking a test. I thought (…) it should be at a level I should be able to do (…) Then it went really, really badly. (…) I got it very wrong. But it felt quite motivating in a way because I really want[ed] to do well (…). [This caused] quite a strong reflection. (…) I should be good enough for this, and [I'm not]. So, I [did] (…) quite a bit of work on [it]. That's the most motivating thing I've done recently.*

The second theme emerged when it became apparent Derwyn felt particularly influenced by Ethan. This was hinted at by the way Derwyn spoke of them working

[2] The NSAA is an admissions assessment for Natural Sciences. It has a mandatory mathematics section, in addition to options in biology, chemistry, and physics.

together when preparing for university admissions assessment. Derwyn subsequently said the following:

> *in my old school (…) I never really approached any [other student] for [help]. (…) when I became friends with Ethan, (…) he had a very strong path [that] he was trying to [follow]. It's that he has strong goals. At the time I wasn't the same. But being [around] someone who actually knows exactly what they want to do. And they're very strong and they're very like capable. It kind of makes you look at yourself and realise that I (…) really needed to (…) produce something. (…) I needed to set my [own goals], and (…) really push myself (…). I felt like having someone there that (…) made [me] accountable also (…) made [me] realise that's (…) what I could be doing too. It (…) [made me] (…) motivated to do better.*

It is clear that Derwyn felt positively influenced by Ethan in a very profound way. By his own description, he was no longer somebody who was not sure what his goals were. Because of Ethan, in the space of one year not only did Derwyn have a strong goal of applying to study in Cambridge. He had pursued it so seriously he was calling his preparation for the admissions test and his performance at the interview his two proudest achievements as a problem-solver. Moreover, Derwyn subsequently secured a conditional offer from the University of Cambridge to study natural sciences. Derwyn characterised the nature of his relationship with Ethan as competitive, describing this competition positively. He said the following:

> *It's always been competitive with Ethan. (…) It [has] always been (…) friendly, you know. [We] try to one up each other, (…) but it's not like we're enemies. It's an enjoyable thing. (…) It makes me feel like, because I'm trying to follow this other person, I'm trying to be accountable to them. I'm trying to do something as good as them. If I'm not doing as [well], then I'm trying to [get] better [at it], so I keep doing it better. I'll watch [times] I've done it better than him and he feels the same way. So, it (…) bounces back. (…) I think it's really like, healthy.*

4.3.5 Summary of Derwyn's Advanced Mathematical-Development

Derwyn's initial fascination with the problems only further intensified throughout his advanced mathematical-development. What began as an eagerness to simply engage with the problems transformed into Derwyn's strong desire to master the art of advanced problem-solving. The way he approached doing so went through an analogous transformation. To start with, he actively tried to work with other gifted mathematicians as often as possible, believing that the more peers he involved, the greater their chances of success would be. Once his advanced mathematical-development was well underway, he no longer took the same view. However, other gifted mathematicians were still significant motivators to Derwyn, just in a new way. He enjoyed collaboration when it led to him having opportunities to think problems through for himself. With their support, he learned that rushing into problems too quickly is not always the best way to make progress. Derwyn eventually became able to realise when his current mindset was hindering his progress, and hence knew when to take breaks to reset it. The closeness he developed with Ethan was particularly significant for his motivation. He subsequently acquired stretching goals for his problem-solving which he sustained a high level of commitment to throughout the whole process. Derwyn's ability to influence and yet also be influenced by others led to his pseudonym. Derwyn, the gifted friend.

4.4 Ethan's Experience of Advanced Mathematical-Development

4.4.1 Ethan's Educational Background

The final participant is referred to as Ethan. He was eighteen at the beginning of the study and is white-British. Ethan lived in the middle east prior to joining the mathematics school, where he attended a private international school. His previous school did not observe the same educational phases as is typical in the UK for students of Ethan's age. In particular, the curriculum was broader, meaning that at sixteen Ethan continued to study many subjects rather than specialising in just three. Hence, despite being the age of a typical Year 13 student, Ethan joined the mathematics school as a Year 12 student. Ethan also lives independently of his parents, who remain in the middle east. Ethan's aspiration was to study natural sciences or physics at university; he applied to the University of Cambridge to read natural sciences.

4.4.2 Ethan's Experiences of Embarking on Advanced Mathematical-Development

The first theme which emerged when it became apparent Ethan thoroughly enjoyed the problem-solving sessions. In particular, he enjoyed having appropriately-challenging problems to tackle, writing often of *'getting stuck in'* to and *'having a go'* at the *'task at hand'*. This enjoyment arose through succeeding in a task that had been selected for him, writing that *'I feel obliged to solve it and therefore enjoy doing it.'*. Ethan therefore perceived the problem-solving as a positive experience within his advanced mathematical-development. He later described what he perceived as the difference with these questions at his subsequent interview, stating that *'I feel like the problems [do not require] you to [have] more knowledge. It's more how good are you at solving [them using] basic knowledge. (…) I immediately [thought they] would improve my problem-solving skills.'*. He enjoyed developing his problem-solving skills very quickly. Writing about a polynomial problem we tackled in the first session, Ethan made the following observation about spotting a pattern in the problem through applying his existing knowledge: *'I think the factorial side of this question was the most enjoyable. Factorials are very useful. (…) [Being taught] to notice patterns helped me to find the factorial in the question'*. However, when Ethan was not able to spot the link between his existing knowledge and a higher-level problem, he described struggling quite commonly. For example, he wrote that *'I struggled quite consistently today in the areas the question was asking about – particularly calculus and coordinate geometry.'*. Both were topics Ethan had already encountered in his A-Level studies; he would generally feel well-acquainted with the relevant principles and methods at that level. However, the problem in question did not use the typical vocabulary. This gave him less opportunity to immediately link the problem with those topics. Ethan reflected on struggling with this question further. He wrote that *'I struggled most with the co-ordinate geometry and the calculus (...) I didn't even know you had to do these things until [Niall] helped me. (…) my knowledge (…) is insufficient to answer these types of questions'*. Struggling to link existing knowledge with advanced mathematical-problems led to him perceiving his current mathematical knowledge and skills as less effective. Hence, his perceptions of his own advanced mathematical-development were impacted negatively by these experiences. However, Ethan's diary entries focused exclusively on how he perceived his mathematical abilities within the current task. Perceiving advanced mathematical-development negatively in the moment did not therefore appear to affect him to a great extent at later times.

The second theme emerged as a result of Ethan writing about several occasions he judged his existing mathematical knowledge and skills negatively. His benchmark was whether he needed help or could succeed alone. For example, he wrote that *'I struggled quite consistently today (…). I needed to ask for help multiple times. I definitely couldn't do the question independently'*. This benchmark is highly suitable for measuring success in A-Level activities, as students ultimately require consistent and independent success to gain sufficient marks for an A* or A grade (2.5.3). Ethan therefore applied his existing notions of success when judging his ability to tackle advanced mathematical-problems. He further reflected that his knowledge and skills felt *'insufficient to answer these types of questions and gain the majority of marks'*. Gaining marks is exactly what mathematicians must do in A-Level examinations to ensure their ultimate success. Ethan subsequently wrote that *'Even with some help, I still felt fairly lost as I didn't know what the question wanted.'*. When asked about what he found helpful during the early stages of the study at interview, Ethan said the following:

> *I think the way we approach [the problems] was pretty good. [You] would give us like a sample of how to do something, and then we would get a similar question and try and solve ourselves. (…) If we really got stuck, we could get help from you at the end. The thing I really liked about [the problem-solving session] was at the end of every question or part way through you would show us how you would do it, and your method was always like an easier way to do it, or most of the time it was so I feel like that definitely helped. And that's something that we don't really do in normal lessons as much.*

Ethan talked of getting samples of the possible methods for a problem from me, then applying those skills independently in a later part of the question or a similar question. However, he also spoke of needing an opportunity to overcome any challenges he faced when solving a problem without my help. By *'really got stuck'* he meant times when he had applied at least a noticeable effort to tackle a problem without making progress. Only once that point had been reached did he perceive the help he obtained from me as particularly supportive. Ethan was therefore describing what he saw as a good balance of help and independence even from the very early stages of the study. Moreover, at this stage, once he had reached the point where help from me was most welcome, he responded positively to methods and approaches being explicitly demonstrated.

4.4.3 Ethan's Maturing Experience of Advanced Mathematical-Development

The first theme that emerged in relation to Ethan's writing about feeling frustration and enjoyment throughout Diary Phase Two. He reflected on the relationship between the two and the subsequent influence on his motivation. Ethan described being *'motivated to get faster and more consistent with them [new questions]'*. He therefore acknowledged he was a developing problem-solver, not yet at the level of speed and consistency he desired and was working towards. However, he usually recounted moments he perceived himself as slow or inconsistent, and hence felt frustration, in a positive way. For example, he stated that *'after doing the questions a few more times slowly and step by step, they've become enjoyable and not frustrating'*. Likewise, Ethan wrote that *'There was a question (…) I got wrong and asked [Kindred] about and he did it quickly and without much difficulty which made me motivated to try and master that kind of question.'*. This became evident in the two other individual emergent-themes arising from Ethan's perspectives from Diary Phase Two. He actively surrounded himself with as many advanced mathematical-problems as possible to ensure opportunities to develop were bountiful. Moreover, Ethan wrote about also feeling a new confidence as an independent problem-solver. However, in times of frustration, Ethan also benefited from interactions with me and the other problem solvers he admired.

The second theme emerged due to Ethan's enthusiastic writing about the problems he was encountering throughout Diary Phase Two. After the first session, he wrote that *'after solving a (or part of a) problem I get more motivated and excited to do more problems if they are difficult and/or interesting questions.'*. Ethan further stated that *'I just felt motivated to solve problems because it was fun doing it'*. He also elaborated what it was about the problems that he was finding it fun to engage with. In particular, Ethan described satisfaction in completing questions in full. By doing so he was witnessing his mastery of the mathematical skills he had been working on. Comments like *'I enjoyed how satisfying it was when it all was expanded and came together to perfectly give us the answer we needed without needing too much extra thinking, it was a very fun question*.' were quite typical for Ethan. He therefore still perceived difficult questions as interesting and consistently sought out as many problems as possible to immerse himself within.

Ethan's earlier hyper-focus on problems posed in the sessions was not as evident in his perspectives from Diary Phase Two. He wrote frequently not only about the problems discussed in the sessions, but also the many he had begun seeking out in his own time from a variety of sources. He wrote that *'Since the last session I have done a British Physics Olympiad*[3] *[BPhO] paper and got almost all of it correct, which I was proud of.'*. For the present purpose, it suffices to know that the level of challenge in the BPhO is similar to the problems I had curated for the sessions in the study (3.9.1, Appendix Five). Ethan also often sought advanced problems in topics he was currently learning in his A-Level studies to work on in his own time. He stated that *'I have been pretty motivated to do (…) matrices questions as we're doing it as a topic right now. (…) I've found [them] (…) very interesting.'*. Ethan still found the questions

[3] The BPhO is a competition pursued by aspiring physicists in the 16-19 phase which challenges them to apply their knowledge of the A-Level Physics curriculum in novel and unfamiliar ways.

posed in problem-solving sessions enjoyable. He wrote that *'I find the problems we do in the sessions very interesting as they're challenging and make you think to try and understand the problem and the quickest way to solve it.'*. Ethan's perceptions of advanced mathematical-development therefore benefitted greatly from exercising the ability he had developed to tackle tough problems on a regular basis, both in the sessions and his own time.

The third theme emerged when it became evident Ethan was referring to how the role of help and independence had evolved. Ethan responded to a prompt in a diary-entry checklist (Appendix Six) which asked him to consider whether social interactions had influenced his perceptions of advanced mathematical-development after a particular session. He subsequently wrote the following: *'There was no single interaction that made me feel motivated or confident'*. An interpretation of Ethan's perspective was unclear in isolation. However, Ethan also reflected regularly on how help from others influenced his confidence. In Diary Phase One, Ethan wrote about interactions (usually between him and me, or him and Derwyn) that had a significant impact. In Diary Phase Two, he described deriving motivation from smaller interactions which supported his independent progress. References to *'tips and tricks'* were common. Ethan wrote that *'I think all the tips and tricks that Niall gives us for the [advanced problems] help a lot and are all important'*. He further elaborated on this in the following diary entry:

> *I got stuck trying to figure out how to do [the question], I find this happens with a lot of [these] questions, sometimes I need to think about it for a while and other times I understand it a lot better and am able to complete it with a bit of guidance. I think this will improve with practice (possibly independent and then go to Niall for help/guidance if I don't understand it)*

Ethan now wrote of needing just *'a bit'* of guidance, rather than his earlier focus on significant interactions with others (4.4.2). Often, this was a small interaction, for example being offered a hint. Other times, it was support to identify a small mistake in his solution. For example, Ethan wrote that:

> *I messed up in the last step with (…) arithmetic (…) [and] didn't end up with the correct answer but managed to get there in the end with Niall pointing it*

out. (…) I was most proud that I was able to do the [question] with almost no help aside from an arithmetic error (…) I think all of the big and small tips we've done are all important and [have] made it possible for me to complete part of the question a lot faster and easier than I would have before.

Ethan was pleased to have finished that particular question with *'almost no help'*. Perceiving a high level of independence when problem-solving was one thing that supported Ethan's positive perceptions of advanced mathematical-development. In the aforementioned diary entry, Ethan was also reflecting on how he had accumulated skills and experience throughout the process of advanced mathematical-development that had facilitated this independence over time. He believed this had led to the ease with which he completed the problem alone on this day. He therefore used his perception of ease as the benchmark against which he judged his confidence in his mathematical skills. Hence, Ethan reported positive perceptions of his advanced mathematical-development when he was able to think through and complete most of a problem independently.

4.4.4 Ethan's Reflections on Advanced Mathematical-Development

The first theme relevant to Ethan is this phase pertained to the other participants. When writing about how they influenced his motivation and progress, Ethan often chose to reflect on his close working relationship with Derwyn. Ethan was subsequently asked about the positive influences of other people at interview. He said that *'Derwyn and I are (…) good friends (…), so it was a lot easier to help each other out and like bounce off each other.'*. Ethan therefore reported having a close relationship as being beneficial for his perceptions of advanced mathematical-development, speaking of helping and getting help from another student being easiest with Derwyn for this reason. Ethan went on to report that he also felt competition between himself and Derwyn. He said that:

> *[A] competition element definitely played a part. (…) with Derwyn, we're (…) competitive towards one another. It's (…) friendly competition that definitely helps. Like you [want to] get better, and whenever there was a hard question where I think we're about similar in like maths and physics skills, we [would] bounce ideas off each other, and it would work quite well.*

Ethan's references to competition with Derwyn were therefore wholly positive. He enjoyed competing with Derwyn, who he perceived as a peer of similar ability, as this led to working closely together on problems.

During his interview, Ethan also mentioned another individual he was influenced by in the classroom, his physics teacher Kindred. There were some earlier mentions of Kindred in his diary entries. However, Ethan did not reflect on the differences between his relationship with Kindred and his relationship with me until his interview. He said that:

> *When [asking] Kindred about physics questions (…) he didn't always know the answer. So, you were sort of working it out with him. (…) But I think whenever I would ask you a question (…) most of the time [you] knew how to do it almost immediately. (…) If I want to know the best way to do it, I come to you. But if I want to work through a question together, I probably go to Kindred.*

Ethan therefore perceived a difference in approach between me and another of his teachers. Moreover, he was able to utilise us effectively by considering the type of help he currently perceived that he needed, and then choosing who he felt was the better teacher to approach on that basis.

The second theme which emerged related to Ethan's homelife. He was unique amongst the participants in that he lived independently. Moreover, his parents lived in the middle east. Ethan made no references to this during his diary entries. However, when asked if anything beyond school had influenced his motivation as a problem solver, he said that *'living alone makes you better at solving problems, I guess, because there are some problems that you have to deal with, that you don't really know how to deal with'*. Ethan also shared that his desire to attend a specialist mathematics school is what led to his decision to return to the UK. Ethan did not share anything more about the support available at home in general. What became clear from this conversation, though, was Ethan's sheer drive and determination to pursue his goals as a problem-solver. It led him overseas to a very different life. He had overcome hurdles like adjusting to living alone while on this quest, and his motivation did not waver when faced with these challenges.

4.4.5 Summary of Ethan's Advanced Mathematical-Development

Ethan's journey as an advanced problem-solver started long before he participated in my study. He knew that he wanted to maximise his potential, believing that studying in a mathematics school (1.5) was his best chance of success. From the beginning he therefore took every opportunity to be challenged. His desire to develop as a problem-solver undoubtedly motivated him to become one of my participants. Right from the first session, Ethan tackled problems enthusiastically. He also knew at this early stage that he would not simply be satisfied with making progress. Rather, he needed to feel an improvement to his independence as a problem-solver. Ethan also expected a lot from himself, not being content unless he left each session understanding every aspect of the problems we had discussed and feeling confident to apply those ideas in the future. This did not, however, stop him from utilising the support available to him. Moreover, he not only understood the type of help he needed to remain motivated, but also knew from whom he could get the right help at the right time. Ethan asked questions eagerly when he needed help, unfazed by asking for help in front of his peers. He responded well to friendly competition, satisfying this need most often by working closely with Derwyn, with whom he became very close. Ethan also consistently surrounded himself with as many opportunities to develop as a problem solver as possible, both in the problem-solving sessions during the study, and in the multitude of problems he pursued in his own time. Throughout it all he held an unwavering focus, was not perturbed by difficulty, and always believed he had the potential to rise to whatever challenges he faced. This led to me choosing the synonym Ethan, meaning strength of wisdom, encapsulating his determination to realise his fullest potential as a problem-solver. He did subsequently secure a conditional offer from the University of Cambridge to read Natural Sciences, as was his goal.

4.5 Summary

This chapter has presented Confur, Derwyn, and Ethan's views in fine detail to showcase the findings of the study from their phenomenological perspectives, and demonstrate how their perceptions evolved over time based on data pertaining to each of the three identified phases. The Discussion and Analysis chapter which follows will demonstrate how these findings were re-analysed to establish the common aspects of advanced mathematical-development the participants described. It sets out the shared themes that arose from the interpretative analysis and critically evaluates the nuances within them in light of the developed theoretical framework and relevant perspectives in the literature.

5 Discussion and Analysis: Adjustment, Feelings, and Relationships in Advanced Mathematical-Development

5.1 Introduction

The Findings chapter presented three detailed perceptions of advanced mathematical-development in answer to the first research question. It remains to establish a wider perspective on advanced mathematical-development to further answer this question, and to utilise this perspective to create a pedagogical model (6.3) for supporting advanced mathematical-development in answer to the second research question. Both are stated again below:

1. How do gifted mathematicians perceive their experiences of advanced mathematical-development throughout the further education phase?
2. What implications do gifted mathematicians' perceptions of advanced mathematical-development have for effective pedagogical approaches which support them through the challenges they associate with this experience?

To answer these questions in the ways described above, this Discussion and Analysis chapter presents the outcomes of the interpretative phase of the analysis. This analysis resulted in three shared themes representing nuances of significant influence within the perceptions of advanced mathematical-development: "Adjustment" (5.2), "Feelings" (5.3), and "Relationships" (5.4). To create these shared themes, the categories that ultimately resulted in the individual emergent themes were reconsidered to evaluate the commonalities and distinctions between how Confur, Derwyn, and Ethan perceived their experiences of advanced mathematical-development (3.12). An example of how categories and codes were repositioned within the shared themes can be reviewed in Appendix Seven. The Discussion and Analysis chapter will also demonstrate how I actively claimed ownership of my role in developing this collective interpretation through using first person language where needed for clarity, and explicitly highlighting the reflexivity considerations I was cognisant of and how they were subsequently challenged (3.3, 3.14). The process of reflexivity is inherently subjective. However, relating findings to independent literature, and in particular applying the theoretical framework (2.6), served to balance the interpretation with objectivity (3.14). Vygotsky's (1978) theory of the ZPD has already been honed for this purpose (2.6). The analysis of findings in this chapter is therefore situated within the Vygotskian theoretical perspective through explicit consideration of the definition of a gifted mathematician's ZPD during their further education and application of the developed vocabulary for describing it (2.6). The intention is twofold, seeking to both facilitate an improved understanding of the nuances of advanced mathematical-development, and to further hone the theory for application to gifted mathematicians during the further education phase in light of the findings.

The first shared theme to be analysed is "Adjustment" (5.2). This section establishes the importance of challenge actually being vicinal to the individual's ZPD for their perceptions of their own advanced mathematical-development to be positive. Furthermore, it details both the benefits to gifted mathematicians when challenged at this level, and the pitfalls they might experience when working beneath or beyond the ZPD, in order that guiding principles can be established to underpin the aspect of pedagogical model concerned with supporting gifted mathematicians throughout this adjustment. The second shared theme is "Feelings" (5.3). Each participant was influenced by how they felt about an experience, causing varying views of their mathematical ability. Their feelings were also a central influence when they judged what was vicinal to their ZPD. This section analyses these feelings, placing particular emphasis on their variable impact and the nuanced perception of frustration and motivation within advanced mathematical-development. The final shared theme is "Relationships" (5.4). Participants' perceptions of their relationships with other gifted mathematicians, including me as their teacher and MKO, influenced how they perceived their own advanced mathematical-development in myriad ways. This section begins by exploring the role of the MKO. An evaluation of the evolving perceived influences of various MKOs within advanced mathematical-development, and how these MKOs differed from each other, follows. Consequently, the implications of the assumptions made about the role of MKO to gifted mathematicians are evaluated. These assumptions are subsequently interrogated, leading to an improved understanding of the role of MKOs within advanced mathematical-development. This is carried forward into an analysis of how the participants related to each other, and how these relationships affected each individual's perceptions of advanced mathematical-development in complex ways.

5.2 Shared Theme: Adjustment

5.2.1 The Abstract Nature of Appropriate Challenge in Advanced Mathematical-Development

The most-strikingly Vygotskian idea that emerged from the interpretative analysis was the concept of appropriate challenge and its necessity for effectively transcending a person's ZPD (Zepeda, Martin & Butler, 2020). All participants wrote about the feeling of being challenged (4.2.2, 4.3.2, 4.4.3) and how this affected their perceptions of advanced mathematical-development. Lynch's (2019) perspective on the nature of appropriate challenge is that social learning models support learners to succeed predominantly through direct demonstration of how to effectively recognise and utilise external resources. This proved difficult to apply directly to the data, which had an emphasis on the challenge of thinking in new and novel ways. Examples of novel thinking skills in the data included those described by Siklos (2019) as necessary within advanced mathematical-development. For example, Ethan reflected on the challenge associated with developing his ability to spot patterns (4.4.2, *ibid.*), and Confur described the challenge in figuring out an effective approach to solving unfamiliar problems (4.2.2, *ibid.*). To apply Lynch's (2019) view would therefore require conceptualising the gifted mathematicians' own thinking as the resource which they were feeling challenged to develop. But thinking is internal. Lynch's (*ibid.*) wider perspective is that utilising tools is a directly-observable process. Hence, people can develop these skills purely through observing the nuances of such a process while it is being practised by somebody who already holds the skills (Smith, 2020). Individuals therefore develop the skills in themselves through a process of mimicry, an imitation which becomes more faithful through ongoing observation, practice, and refinement (Braddon-Mitchell, 2019). However, this is usually in relation to skills which are tangibly observable in a way that honing one's own thinking is not. The challenge faced by the participants, then, surrounded how to develop problem-solving skills through honing their own metacognitive thinking, by mimicking an abstract process (Villani *et al.*, 2019) that even a person skilled in that thinking would not adequately demonstrate purely by token of practising it. My professional experience of the complexity of this process, specifically in relation to problems throughout advanced mathematical-development, is informed by GMI's (2019) perspective on modelling metacognitive skills to mathematicians more generally: abstract problem-solving skills cannot be 'seen'; they must therefore be actively demonstrated via another means if an individual is to 'observe' all their nuances by being in the same space as somebody actively practising them (*ibid.*). The present analysis therefore has three aims. Firstly, to consider the challenge of honing the abstract skills associated with metacognitive problem-solving (Villani *et al.*, 2019) vicinal to the ZPD. Secondly, to better understand the challenging experience identified by Halmo, Yamini, and Stanton (2024), referred to as 'metacognitive discomfort', as it pertains specifically to gifted mathematicians undertaking advanced mathematical-development. This difficulty is experienced when an individual is adjusting to the more-abstract challenge of honing metacognitive skills, and subsequently impedes metacognition development (*ibid.*). The third aim is to identify an effective means through which metacognitive problem-solving processes can be actively demonstrated (GMI, 2019) to gifted mathematicians undertaking advanced mathematical-development. Hence, the critical evaluation of the nature of adjustment in the specific context of advanced mathematical-development that follows (5.2.2) subsequently motivates effective strategies practitioners can utilise to support gifted mathematicians through the adjustment (5.2.3).

5.2.2 Subtheme: Adjusting to Appropriate Challenge

All participants described the advanced problems they were considering in ways that suggests they were vicinal to their ZPD. However, they each experienced the associated difficulties with adjusting to this challenge in different ways. Starja, Nikolova, and Shyti (2019) found that some gifted mathematicians benefit from a natural ability to persevere with struggle, and hence are able to work smarter rather than harder when working vicinal to their ZPD. This suggests that some gifted mathematicians are instinctively able to recognise when they are feeling negatively about their rate of progress; hence, they can quickly acknowledge that their current method is not working. Such gifted mathematicians are therefore able to identify ineffective methods and subsequently abandon them, adopting different methods with greater ease. The conceptualisation of Starja, Nikolova, and Shyti's (2019) perspective guides an interpretation of why Derwyn's experience was different to both Confur's and Ethan's in this regard. Derwyn stated he derived motivation from *'not being discouraged'* when the questions felt hard, which led him to try alternative approaches more naturally. Contrastingly, both Confur and Ethan's perspectives in relation to adjustment demonstrated that neither benefitted from a natural affinity which enabled them to remain motivated when experiencing challenge in a frustrating way (*ibid.*). Confur's comments about *'banging my head against a brick wall'* when describing this challenge suggests he was determined to keep trying in vain, so was working *'harder'* rather than *'smarter'* (*ibid.*). Similarly, Ethan talked of this being a *'considerable struggle'*. Having always made expedient progress in activities beneath his ZPD with which he was accustomed, he judged himself harshly and therefore initially tried to transcend his ZPD through sheer force of will during Diary Phase One. Where Derwyn was predisposed to the flexibility of thinking Starja, Nikolova, and Shyti (*ibid.*) described, Ethan and Confur were not. Hence, Derwyn was able to benefit from his advantageous predisposition.

Erdogan and Yemenli (2019) described gifted mathematicians with a variety of attitudes, both helpful and unhelpful, towards being appropriately challenged. Whether the relative ease or difficulty in which the three participants adjusted to appropriate challenge can be attributed to their diverging attitudes (*ibid.*) or predispositions (Starja, Nikolova & Shyti, 2019) is unclear. However, when applied to the participants' perspectives, the literature raises two considerations for practitioners supporting gifted mathematicians through the early stages of advanced mathematical-development. Firstly, that gifted mathematicians fall into two groups (Erdogan & Yemenli, 2019): those who experience adjustment to working vicinal to their ZPD when pursuing advanced mathematical-development as an uncomfortable challenge (Halmo, Yamini & Stanton, 2024); and those who experience it with greater ease (Starja, Nikolova & Shyti, 2019). Secondly, that gifted mathematicians falling into these distinct groups are likely to require a different balance of support relating to cognitive and affective development (Brigandi *et al.*, 2018) throughout the adjustment.

Both Confur and Ethan eventually found a way through their initial period of adjustment, albeit with more difficulty than Derwyn. Rubtsov's (2020) assertion that ZPD progression is usually gradual in nature, only feeling significant when viewed in retrospect, facilitates an apposite interpretation both of Confur's lived experience and his hindsight reflection. He said that after feeling he was *'banging his head against a brick wall'*, it *'all of a sudden gives way'* and *'then your head doesn't hurt anymore'*. His determination to keep trying therefore eventually won out. Hence, at later stages of advanced mathematical-development he was able to perceive challenge vicinal to his ZPD in a substantively different way, through reflecting on each time he had been stuck but had eventually succeeded. Parish's (2018, pp. 3-5) research into self-limiting mindsets identified four qualities of particular relevance to the participants: persistence, perseverance, drive, and grit. Derwyn, being naturally able to identify when methods were not fruitful and hence change them in order to succeed, naturally possessed perseverance. Confur and Ethan, contrastingly, came into the study with persistence. What they lacked initially was the flexibility of mind to adapt their unsuccessful methods (Starja, Nikolova & Shyti, 2019). However, through grit (Parish, 2018) in Ethan's case, and drive (*ibid.*) in Confur's, both were able to maximise their persistence, and subsequently develop perseverance. Ethan was strongly motivated by his desire to pursue elite university admission (4.4.1, 4.4.5) in physics (see 5.4.4). To succeed in that higher endeavour, he therefore needed to score well in the NSAA, which in turn required him to develop the problem-solving skills associated with advanced mathematical-development. Confur, contrastingly, was motivated by an internal desire to become as skilled and knowledgeable a mathematician as possible, independently of other people (see 5.4.5). For Confur and Ethan, either grit or drive (*ibid.*) therefore became the source of motivation to persist for longer during times when working vicinal to their ZPD felt over-challenging. Neither therefore perceived their lived experience of failure as indicating they could not ultimately succeed. In turn, they were able to benefit from their experiences during Diary Phase One, through which their persistence evolved into perseverance. The current analysis has three outcomes. Firstly, that gifted mathematicians who find the process of adjustment challenging might need support to develop perseverance. Secondly, that moments of struggle are ideal opportunities for gifted mathematicians to think about changing their method, and hence develop perseverance faster. Thirdly, that a gifted mathematician's reason for pursuing advanced mathematical-development is an important factor in understanding their underlying motivation, through which they can develop perseverance. However, it still remains to critically evaluate precisely how to support their development in these ways (5.2.3)

Olszewski-Kubilius and Corwith's (2021) study investigated the need to support gifted people to plan appropriate activities over a longer term. Their (*ibid.*) view is that setting gifted people up to perceive their development as the product of working over an extended period limits the adverse effects of momentary lived experiences of failure. Instead, their many successes can be emphasised, enabling gifted people to see that failure in a given moment does not lead to failure overall (*ibid.*). Encouraging gifted mathematicians to focus less on the current moment and instead reflect on their progress over time would therefore help them perceive their advanced mathematical-development more positively (*ibid.*). However, gifted mathematicians have no successful experiences of working vicinal to their ZPD at the onset of advanced mathematical-development, only acquiring them through their sustained efforts. Hence, the concept of reflecting on success in hindsight has no value at the earliest stages of adjusting to advanced mathematical-development. Moreover, Agaliotis and Kalyva (2019) found that underachieving gifted people generally find it more difficult to sustain motivation than typically-developing peers. This suggests that the support offered to Confur and Ethan during Diary Phase One (3.9.2) helped sustain their motivation sufficiently to overcome this challenge. Brigandi *et al.*'s (2018) findings add further perspective to the possible advantageous features of the support offered throughout the study which helped Confur and Ethan to adjust successfully. They (*ibid.*) found that gifted students benefit from homogenous grouping with like-minded peers, and from teachers specialising in gifted education. In their study (*ibid.*), these factors were positive influences on both cognitive and affective development. Hence, as Confur and Ethan partook of sessions exclusively formed of gifted mathematicians (3.8) and led by a specialist educator (1.5) during Diary Phase One, their adjustment could have been supported in two ways. Firstly, through their affective development (*ibid.*), which honed their ability to regulate their feelings in relation to experiencing adjustment to advanced mathematical-development. Secondly, through their cognitive development (*ibid.*), which honed the metacognitive skills associated with appropriate challenge directly (5.2.1). While suggesting apposite avenues to explore in relation to developing pedagogies to support gifted mathematicians with their adjustment to appropriate challenge, Brigandi *et al.*'s (2018) study does not detail any strategies for supporting gifted mathematicians with advanced mathematical-development specifically. The conclusion of this analysis of adjustment that follows therefore addresses the second research question (5.1) by critically evaluating pedagogies pertinent to supporting gifted mathematicians through their adjustments to advanced mathematical-development in light of the data and literature.

5.2.3 Subtheme: Support with Adjustment to Appropriate Challenge

Derwyn perceived his experience of adjusting to appropriate challenge in a less-frustrating way than his fellow participants (5.2.2). However, that is not to say that frustration only hinders gifted mathematicians in their pursuit of advanced mathematical-development. The nuanced perceptions of frustration and motivation are the subject of later analysis (5.3.3). For the present purpose, it suffices to focus on how to improve the experience for those gifted mathematicians, like Confur and Ethan, who find it challenging to process negative emotions into positive feelings during the early stages of advanced mathematical-development. The studies of

Mofield and Parker Peters (2018, 2019) identified a perfectionist mindset as common in gifted people. Specifically, gifted people are more likely to hold a perfectionist mindsight than their typically-developing peers (Mofield & Parker Peters, 2018); this mindset leads them to interpret the first difficulty they experience as a major failure which destroys their pre-existing perception of self-perfectionism (Mofield & Parker Peters, 2019). This was further elucidated by Akkaya, Dogan, and Tosik (2021), who found that perfectionism levels in gifted students increase as they progress through schooling initially, but then declines in the later stages of education. Moreover, it has been found that a gifted person's positive perception of perfectionism declines throughout education, while their negative perception of perfectionism increases (Kahraman & Bedük, 2016). A synthesis of these findings facilitates an apposite interpretation of Ethan's experience. Ethan held himself to a perfectionist standard initially. In the early stages of advanced mathematical-development, he expected to succeed with the advanced problems to the same extent he succeeded with A-Level problems (4.4.2). In particular, he expected to be able to do them completely and independently. Consistently living up to his perfectionist standard in activities beneath his ZPD was therefore a significant source of positivity prior to engaging in advanced mathematical development. However, like gifted students at later stages of education in the aforementioned studies, upon pursuing advanced mathematical-development Ethan progressed past the point where perfectionism was sustainable. The destruction of his perceived perfectionism subsequently affected his ability to rise to challenge which was actually vicinal to his ZPD. However, although he felt frustration when faced with the inevitability of his own imperfection initially, Ethan at some point stopped focusing simply on the answers and his ability or inability to solve problems perfectly. Instead, he involved himself more with mastering the associated problem-solving skills (4.3.3). Rice and Ray (2018) identified that those gifted people who are less naturally-resilient are at greater risk of stress-related difficulties when adjusting to their own imperfection. This stress is then destructive for further giftedness development (*ibid.*). This is not consistent with the accounts of adjustment in the data. Ethan (4.4.3), and indeed the other participants (4.2.3, 4.3.3), actually reflected positively on their earlier feelings of frustration. This is also contrary to the perspectives of Greenspon (2021) and Grugan *et al.* (2021), that failing to meet the impossibly-high standards associated with perfectionism is destructive for self-worth. Although this might have been the lived experience of the participants, it did not stop them from pursuing advanced mathematical-development altogether. Rather, through persevering long enough to experience independent success, their hindsight reflection on the challenging moments had a positive impact at later times. The negative moments served to intensify their later experiences of success, resulting in euphoric moments of triumph which instilled motivation to transcend their ZPDs; this more than outweighed the difficult moments in hindsight. Czarnocha and Baker (2021) described the importance of similar eureka moments when mastering a mathematical concept or skill after earlier struggle. Although their (*ibid.*) perspective was not specific to gifted mathematicians, it is illuminative for the participants' experiences. This suggests that, when supporting gifted mathematicians through their adjustment to appropriate challenge, the goal should not be to avoid frustration or other negative feelings altogether. Instead, effective support should enable gifted mathematicians to experience negative feelings when working vicinal to their ZPD initially. However, they should be sufficiently encouraged to persist when emotions are initially felt negatively, and hence develop the necessary perseverance (5.2.2) to process them into positive feelings, rather than seek to remove negative emotions altogether. This subsequently became a core principle of the pedagogical model (6.3).

Prior analysis (5.2.1) concluded that appropriate-challenge in advanced mathematical development is metacognitive, and hence abstract, in nature. Both Sternberg (2018b) and Shore (2021) described developing metacognitive practices as a mechanism through which gifted learners can transcend their ZPDs to realise their maximum potential. Shore (*ibid.*) in particular described metacognition as the means through which gifted people can think about their goals and hence determine a means of progressing towards them in a specific context. In the study, the goal might be interpreted as a participant's desire to develop as a mathematical problem-solver. However, although Confur, Derwyn, and Ethan seemingly shared this aspiration, it was not their primary objective. Instead, it was motivated by their variety of higher goals (4.2.5, 4.3.5, 4.4.5), within which pursuing this aspiration was just one aspect. Moreover, the metacognitive approaches were not just the means of thinking about the goal abstractly, but also a skill of effective mathematical problem-solvers the partipants were actively working to acquire. Turgut and Uğurlu (2024) established that metacognitive development has an indirect positive impact on mathematical resilience for gifted learners. Metacognitive development is therefore especially beneficial for gifted mathematicians during their further education, for whom it has also been argued (2.5.3) struggle due to a lack of mathematical resilience (Kerr, 2021) when first feeling genuinely challenged by mathematics. By the further education phase, a typical gifted mathematician has succeeded extensively in learning mathematical methods via their steps being detailed verbally, and is accustomed to deriving positivity from succeeding after being supported in this way (Gottfried, 2019; Beek *et al.*, 2017). This perspective suggests that simply detailing each of the specific steps in a particular method verbally, that they might be observed more readily by gifted mathematicians, would not support their metacognitive development. In particular, because it neither adequately demonstrates the metacognitive problem-solving skills, nor requires the learner to actually practise them. Confur struggled to even comprehend the methods when demonstrated by me during the early stages of advanced mathematical-development. Relating those methods to his wider development as a mathematical problem-solver did therefore not take place until the scaffolding strategies had moved on throughout the study. It is also particularly relevant that the prior analysis (5.2.1) also determined that simply practising metacognitive problem-solving skills is insufficient to fully exposit their nuances. Subsequently, gifted mathematicians would therefore find it difficult to develop metacognitive problem-solving skills through observation and faithful mimicry. Hence, it is important that scaffolding techniques demonstrate the nuances of the necessary abstract thinking-skills more explicitly via another means.

Avhustiuk, Pasichnyk and Kalamazh (2018) identified a pitfall they termed ‘the illusion of knowing’, which arises when an individual mistakenly perceives a specific instance of cognitive development as metacognitive development. This leads them to believe they have developed skills they can apply more widely when, in reality, they merely developed knowledge specific, and hence solely applicable to, the problem under consideration (*ibid.*). In the study, this illusion could have arisen from the participants learning how to solve a single problem through focusing on the information shared about the relevant methods in particular (a single instance of cognitive development), rather than developing their skills as advanced problem-solvers in general (overarching metacognitive development). Should the MKO simply solve a problem live with full detail of the method, they would entirely deprive gifted mathematicians of metacognitive development in two ways. Firstly, by assuming they are entirely unable to transcend their ZPD without an MKO's help. This was addressed by both Derwyn (4.3.2) and Ethan (4.4.2), who reflected negatively on getting too much help too soon when persevering with struggle. Secondly, the gifted mathematicians would be set up to focus on learning methods specific to the problem being considered over developing wider problem-solving skills, further instilling the illusion of knowing described above (*ibid*). Ahn, Hu, and Vega (2020) posited that MKOs might be considered role models, and hence that learners would do better to imitate MKOs' behaviours than simply follow MKOs' words (*ibid.*). Conceptualising metacognitive problem-solving as a habit or behaviour of thinking, which subsequently results in the conception of a particular method, is therefore one means of supporting the metacognitive processes to be more observable (*ibid.*) to gifted mathematicians. This is not to say that metacognitive problem-solving cannot be modelled in detail verbally. Rather, that the topic of an MKO's commentary should be the processes they go through when discerning or designing a method to a problem, not just the method specific to that problem. This becomes the basis of a refined scaffolding strategy of highest intensity within the pedagogical model (6.3).

5.3 Shared Theme: Feelings

5.3.1 Emotions and Feelings in Advanced Mathematical-Development

This analysis of the nuances of the second shared theme, pertaining to the feelings of the participants within their advanced mathematical-development, will draw a critical distinction between feelings and emotions in the following way. As subconscious acknowledgements of external stimuli, emotions are the raw data associated with an experience (Allyn, 2022). Feelings, then, are the way emotions are subsequently processed and presented to the conscious mind (Farnsworth, 2020). Kerr's (2021) perspective is that gifted people need support to manage their instinctive response to negative emotions which arise when presented with evidence of their own imperfection for the first time (2.5.3). While this view appeared to directly pertain to Confur and Ethan whose feelings hindered them when meeting appropriate challenge, it is less applicable to Derwyn. As such, throughout this analysis there will be many examples of the participants experiencing similar difficulties and successes, which under the above conceptualisations are interpreted as causing similar emotions. However, the way they perceived these experiences varied greatly; hence, it is their feelings which differed and, subsequently, had different impacts on each individual's perceptions of their ability to make progress. The distinction between emotions and feelings within this analysis not only supports clear descriptions of the participants' perceptions of advanced mathematical-development in answer to research question one (5.1); it also provides another reason to believe gifted mathematicians can overcome the challenges they perceive in their feelings towards advanced mathematical-development. Specifically, it implies the possibility that gifted mathematicians can learn to process their emotions in a more-positive way, and hence create positive feelings from them which benefit their subsequent development. This consideration within the analysis forms the basis for the aspect of the subsequent pedagogical model (6.3) concerned with supporting gifted mathematicians to change their feelings within advanced mathematical-development, and hence to answer research question two (5.1).

5.3.2 Subtheme: The Impact of Negative Feelings

During Diary Phase One, Confur wrote of overriding feelings associated with his perceived failure with a given problem (4.2.2). Grugan *et al.*'s (2021) finding that negative experiences are detrimental not just to a gifted person's lived experience of it in that particular moment, but in a more longstanding way, is illuminative for Confur's experience. He described this as affecting him beyond the current moment. However, this was not the case for Ethan. When asked about negative experiences that he believed were significant, he said that he *'could not remember a specific one'*. He also never made references to earlier negative experiences in his diary entries when recording his perceptions at a specific time, only ever reflecting on the current activity. Ethan did not state he had perceived experiences which led to negative feelings detrimentally at later times. This therefore contradicts Grugan *et al.*'s (*ibid.*) finding that negative emotions create long-lasting difficulty. However, the existence of two contradictory perspectives on a similar nuance of the experience of advanced mathematical-development is neither surprising nor concerning. Snyder and Wormington (2020) described two common misconceptions: that all gifted people are motivated, and that this motivation remains consistently high throughout education. The divergence in Confur and Ethan's perspectives of how negative emotions affected them over time is therefore well explained by this variety of evolving levels of motivation in gifted people (*ibid.*), and hence was anticipated. Moreover, both ultimately described having overcome earlier challenges and feeling successful at later times. Greenspon's (2021) view that some gifted learners need support to 'move beyond' negative self-judgement, where others are either able to do so without assistance or not judge themselves negatively in the first place, further informs the difference between Derwyn and the other participants in this regard. Derwyn experienced the negative aspects of adjustment to a lesser extent than his peers, and hence did not require significant support to overcome them (5.2.2). However, neither Confur nor Ethan's lived experiences of adjustment were so destructive they completely derailed their commitment to advanced mathematical-development. Their judgements of their own successes evolved positively as they persevered (see 5.2.2). In light of Greenspon's (*ibid.*) view, that neither Confur nor Ethan found adjustment straightforward, but went on to persevere and hence succeed anyway, suggests that the support in place helped them learn how to process negative emotions into positive feelings. Moreover, Vygotskian ideas that ZPDs are specific to the individual, and are transcended at different rates at different times (Lerman, 2019), further suggests that such theories are particularly apposite for conceptualising advanced mathematical-development in light of this variety being evident within the participants' perspectives. Svendsen and Burner (2023) found that gifted learners become less stressed and more motivated when supported to sustain their engagement with the learning process causing negative emotions. The findings therefore suggest that assessment practices utilised by practitioners supporting advanced mathematical-development should include formative assessment which explores how each gifted mathematician is *feeling* about their progress, and how their feelings are affecting their perception of what is vicinal to their ZPD. This then highlights which gifted mathematicians need particular support to remain engaged when experiencing a negative emotion (*ibid.*).

Where Confur and Ethan's experiences of negative emotions usually hindered their advanced mathematical-development (4.2, 4.4), Derwyn's perception of negative emotions was not as straightforward (4.3). He did reflect often on feeling stuck and frustrated. However, where the negative emotions associated with these experiences were typically inhibiting for his peers, he was not only able to create positive feelings in the presence of negative emotions, but did so most of the time (4.3.2, 4.3.3). Derwyn perceived frustration as a sign meaningful progress was close by. Having always viewed feeling challenged by mathematics as an exciting opportunity to learn and develop my skills, I identify strongly with Derwyn's experience (1.5). For the present analysis, it is notable that Derwyn and I seemingly naturally shared an attitude at the onset of advanced mathematical-development that Siklos (2019) argued actually should be the end goal. He (*ibid.*) reasoned that although not knowing what do when faced with a novel problem can be disconcerting, through practice this can begin to feel like an enjoyable sense of puzzlement. Starja, Nikolova, and Shyti (2019) noted specific examples of gifted mathematicians overcoming challenge quickly. These individuals are like Derwyn in that they were unfazed in spite of challenge. Likewise, Parish (2018) took the view that some of the gifted mathematicians she researched benefitted from a belief in their own ability to make progress despite challenge. Her (*ibid.*) view, that gifted mathematicians progress more-easily when they perceive obstacles as opportunities to develop, is one apposite interpretation of Derwyn's perspective on this issue. This explains why what she (*ibid.*) would call Derwyn's 'non-limiting mindset' better facilitated his progress when he felt challenged. Mun and Hertzog (2018) distinguished the feeling of confidence in existing mathematical knowledge and skills from a more-nuanced belief in an ability to make mathematical progress, documenting mathematicians holding both modes of perception. Both Confur and Ethan found it initially difficult when they perceived their rate of progress with a challenging problem as too slow, judging their ZPDs to have subceeded when realising their existing knowledge and skills were insufficient to accelerate that progress. Contrastingly, Derwyn did feel momentarily like his knowledge and skills were insufficient. However, he still believed he could, and indeed would, transcend his ZPD by developing the necessary skills. This is not to say he made any assumptions about how quickly that transcendence would occur, nor that he never felt frustration negatively. Rather, he perceived within his frustration not only an opportunity to develop, but both the motivation to pursue that opportunity and inherent belief he would rise to the challenge. The findings therefore directly corroborate Mun and Hertzog's (*ibid.*) distinction between two of the prevailing feelings caused by negative emotions. Moreover, Talsma *et al.*'s (2018) finding that self-belief is the foundation upon which individuals build positive perceptions of their academic pursuits, underlines the importance of the findings. This analysis therefore continues with an exploration of the nuances of negative emotions and, in particular, how gifted mathematicians can be supported to learn new ways of processing them into positive feelings throughout advanced mathematical-development (*ibid.*) specifically.

5.3.3 Subtheme: Frustration Becoming Motivation

Derwyn's nuanced perception of frustration and motivation was perhaps most evident when he reflected on his preparation for the NSAA, which he began after Diary Phase

Two. Derwyn spoke of his early experiences of NSAA preparation (4.3.4). Although he perceived this as challenging, the challenge did not undermine his belief in his own capability. What he was actually able to achieve, as an independent problem-solver (Vygotsky, 1978), or "unaided" (2.6) was therefore below that which he anticipated. Hence, this experience could only have subceeded Derwyn's ZPD under both definitions. He was nonetheless able to remain motivated to successfully rise to the challenge, which might therefore be considered paradoxical. However, although Wilson and Janes (2008) found that constructivist learning approaches have the power to influence positive perceptions of the learning experience, they also held that these perceptions exist independently of academic development. Derwyn perceived his ZPD to have been subceeded in this instance. This should not, therefore, be conflated with the idea that such an experience must necessarily have damaged his perceptions of his potential to pursue advanced mathematical-development further (*ibid.*). His unwavering underlying belief was that he could and would rise to the challenge, despite a negative lived experience of that challenge. Hence, he was able to simply accept that his ZPD had been subceeded slightly. This allowed him to reposition himself within it, and, subsequently, he was able to build upon what he did feel confident to do to transcend his ZPD incrementally. While gifted mathematicians can be hindered by a negative judgement of what is vicinal to their ZPD, they therefore do not have to be, depending on their underlying beliefs (*ibid.*).

Derwyn held a positive outlook towards making progress in times of challenge at all stages of the study. Contrastingly, Confur and Ethan developed towards this attitude over time, but did not hold it initially (2.4.2). During Diary Phase Two, Ethan in particular wrote specifically about frustration leading to enjoyment and motivation when not finding the problems immediately straightforward to solve. Mentz and Lubbe (2021) wrote of one driver of an individual's motivation to continually self-improve: their evidence of having made progress through ipsative assessment. The benchmarks by which Ethan judged his advanced mathematical-development were originally ipsative in nature in that, perhaps subconsciously, he compared his current performance to his previous performance (*ibid.*). Prior ability was therefore a benchmark against which Ethan judged what was currently vicinal to his ZPD. In particular, he based this judgement on the speed and consistency with which he was completing the problems (4.4.2, 4.4.3). It might therefore have been expected that Ethan would feel frustrated when he was not consistent or fast enough at later stages in the study. However, this was not evident within his perspectives from Diary Phase Two. As a specific example, Ethan wrote about struggling with a question in an A-Level Physics lesson during this time (4.4.3). Had Ethan applied his earlier notions of success and failure in this moment, he would have judged himself harshly given that it was at a lower level (Vygotsky, 1978) than questions he was already solving quickly and consistently; his ZPD would have been subceeded by this experience. At earlier times, Ethan did reflect on his perceptions of advanced mathematical-development being impacted negatively by such an experience. However, by Diary Phase Two he began perceiving times he felt frustrated as opportunities to improve in the same way Derwyn was able to at all stages of the study. In this way, Ethan used earlier experiences of negative emotions to subsequently focus similar emotions into positive feelings at later stages of advanced mathematical-development.

Ethan's diary entries appeared to explicitly demonstrate his ZPD had transcended. Contrastingly, Confur's evolution in this regard was somewhat subtler. He described feelings stemming from lived experiences often having an ongoing impact at later times (4.2.2), and this was evident during all stages of the study. However, the longstanding nature of these feelings did not ultimately hinder his ZPD transcendence altogether. The view of Eun (2019), McLeod (2024b), and Nardo (2021) is that, although a given moment of intense feeling can have a significant impact on the individual, longer-term development is the result of a collection of such moments. Hence, ZPD transcendence is gradual (McLeod, 2024b; Eun, 2019), and its speed depends on the proportion of these experiences which are felt positively (Nardo, 2021). Confur's progress with advanced mathematical-development was therefore benefitting from his ever-increasing proportion of positive experiences over time. This therefore not only suggests that he made significant progress throughout the study, but also that his rate of progress was accelerating. This is further evidenced by his response when asked about his most-positive experience of problem-solving at interview. He cited being most excited to complete a timed MAT assessment and score above the mark threshold (4.2.4). Hence, the suggestion for accelerating progress for gifted mathematicians like Confur, whose feelings exert a significant ongoing impact, is to curate experiences which lead to positive feelings as early as possible. This is subsequently reflected in the developed pedagogical model (6.3).

5.4 Shared Theme: Relationships

5.4.1 The Status of the More Knowledgeable Other

Vygotksy's original definition of the ZPD makes reference to 'problem solving under adult guidance or in collaboration with more capable peers' (Vygotsky, 1978, p. 86). Vygotsky conceptualises a person's potential under this definition as what they are able to do with appropriate help from another person. This reflects the social nature of his theory (McLeod, 2024b). Barrs (2021) posited that, when Vygotsky formulated the ideas that would subsequently form the basis of Mind in Society, his seminal work which put forward the basis of his sociocultural theory, he reflected not only on his experiences as an educational psychologist and researcher of psychology, but also as a teacher. This may explain why the role of others has always been so prominent within Vygotsky's conceptualisation of the ZPD (Tudge, 1992). The role of support from MKOs is therefore suggested to be an essential component within the ZPD (McLeod, 2024a). The MKO was acknowledged in the definition of the ZPD adapted for application to gifted mathematicians as "another person who is highly skilled in the task" (2.6). This explicitly acknowledged that the skills gifted mathematicians might seek to develop throughout their advanced mathematical-development are necessarily of a higher level than typical (2.5.3). Barrs (2021) also put forward that the power within Vygotsky's theory lies not only in MKOs who hold a genuine desire to help other people, but those with the correct skills. This analysis of relationships therefore proceeds by critically evaluating how I, as the MKO, supported or hindered each participant's ZPD transcendence throughout both diary phases.

5.4.2 Subtheme: Relating Through Scaffolding Strategies During Diary Phase One

Confur initially had a negative perception of the scaffolding strategies employed during Diary Phase One. Immediately following the first session, he wrote: *'I fell behind (…) and have consistently struggled on understanding.'*. Confur was referencing the speed of his own understanding during the modelling element of the problem-solving sessions. See 3.9.2 for details of the scaffolding strategies utilised. Confur's sense that the strategies being employed had been deliberately designed to help him make progress was therefore present when determining his 'level of potential development through problem-solving under adult guidance' (Vygotsky, 1978, p. 86). Confur did not, therefore, *initially* perceive his propensity for transcending his ZPD in the positive way Hedegaard (1992) proposed, as his inherently-reachable potential. Instead, in the early stages, feeling struggle despite being aware his development was being supported influenced his "current perception" of the task, which he judged "unfeasible even with support from another person who is highly skilled" (2.6). This finding therefore further supports two aspects of the adapted definition of the ZPD. Firstly, the emphasis on a gifted mathematician's own current perception as the mediator for judging what aspects of advanced mathematical-development are vicinal to the ZPD. Secondly, the positioning of what feels impossible (beyond) before what they can confidently do (beneath), within that current perception.

Bates' (2023) perspective is that a negative response to scaffolding strategies often indicates the scaffolding requires further tailoring to be effective for the individual. This informed another interpretation of this aspect of Confur's perspective. In particular, his statement was made in relation to a scaffolding strategy being utilised concurrently with all participants. Puntambekar (2022) referred to using the same strategy to simultaneously support multiple learners as distributed scaffolding. She (*ibid.*) found that for such approaches to successfully support multiple learners, it needs to be possible for them to benefit from scaffolds flexibly so that each can individualise how their development is facilitated. This suggests that, perhaps unconsciously, learners tend towards the scaffolding strategies most beneficial for them in that moment. It follows in turn that the approach to supporting the initial phases of advanced mathematical-development (3.9.2), which utilised multiple scaffolding techniques and hence offered options to each gifted mathematician, is indeed more beneficial than using any technique in isolation. However, during times when a single technique was being employed as distributed scaffolding (*ibid.*), individualised scaffolding could have been more beneficial (Kim, Belland & Axelrod, 2018). For Confur, individualisation during the initial stages of advanced mathematical-development might have helped him avoid perceiving it negatively in general, simply because the distributed scaffolding technique was ineffective for him in that moment.

Writing at a later stage of Diary Phase One in relation to *succeeding* whilst demonstration was employed as the scaffolding technique, Confur stated: *'I am proud of my understanding and the speed at which I got it. I didn't feel like I particularly slowed down at any point.'*. Eun (2019) wrote of the role of the individual's own voice in determining what is proximal within their perception of their own ZPD. Confur's inner dialogue had shifted positively. This was therefore potentially indicative that his ZPD had transcended. However, Eun's (2019) perspective can also be applied to interpret this in the converse sense; it is possible that the positive or negative feeling within Confur's inner dialogue on each occasion is what influenced his judgement of his ZPD, not vice versa. Abtahi's (2018) view of Vygotsky's work applied as a theory of mathematical development describes assessment of progress as ipsative, and hence is pertinent to the analysis of this uncertainty. Confur's lived experience of demonstration as a scaffolding technique was negative initially. This facilitated his later self-comparison. Hindsight reflection therefore enabled him to perceive this positively at later times, serving as a tangible example of overcoming the earlier hurdle. Abtahi's (*ibid.*) perspective therefore suggests that Confur's improved perception of advanced mathematical-development relied upon him directly comparing his current success with less-successful earlier experiences he perceived negatively at the time. This further justifies the emphasis on what is currently perceived as “beyond” before “beneath” within the adapted definition of ZPD (2.6). Moreover, it supports my interpretation based on my professional experience, that accomplishments within advanced mathematical-development *feel* significant precisely because the endeavour was initially perceived as difficult or uncomfortable; hence, that, despite being described through words usually interpreted negatively, a feeling of genuine challenge can be motivational for gifted mathematicians, as it was for me (1.5). This provides further evidence for the nuanced suggestion that, although gifted mathematicians might need more individualised support than was offered during Diary Phase One of the study, their lived experience of conceptual difficulty should not be excised altogether during the early stages of advanced mathematical-development.

The conflation of scaffolding with the wider concept of ZPD has been critiqued (Xi & Lantolf, 2021) in relation to supporting the development of higher cognitive processes. This is because the effectiveness of scaffolding techniques relies on the individual’s ability to react intelligently to these stimuli, whereas the ZPD can be progressed through all manner of social processes, intentional or otherwise (*ibid.*). As gifted mathematicians with high existing levels of mathematical knowledge and skill (2.5.3, 3.8), both Confur and Ethan had the capacity to progress, and prior experiences of progressing, their advanced mathematical-development through both reacting to intentional scaffolding and via unplanned social mechanisms. However, Xi and Lantolf‘s (*ibid.*) perspective remains relevant, offering one interpretation of the difference between Confur and Ethan. It was their *reaction* to negative lived experience that distinguished them. Both Confur’s introspection and Ethan’s instinct to ask questions might be interpreted as their respective means of engineering a different social process (*ibid.*) through which to support their advanced mathematical-development when the intentional scaffolding technique proved unhelpful. This was not a conscious process on their part in the same way a teacher might purposefully design and implement scaffolds to support a learner’s development in the classroom (Margolis, 2020). They were not even aware that a Vygotskian lens was to be applied analytically to their perspectives (2.6), and hence could not have thought directly of their ZPD nor the social processes needed to transcend it in those terms. Moreover, neither did the alternative social processes they actually tended towards during this phase of the study prove to be any more effective in supporting ZPD transcendence than the demonstration scaffolding being employed. Both reflected negatively on this moment, demonstrating that they did not perceive their respective reactions as helpful.

During the first session of problem-solving during the study (Appendix 2), I was influenced by Radmehr and Drake's (2018) conclusion that carefully designed questions, and carefully constructed answers to questions, can be leveraged as scaffolding techniques (3.9.2) in their own right when supporting mathematical problem-solving. I therefore interpreted Ethan’s regular questions as insights into his individual support needs, and adapted my approach through selecting words carefully when answering (*ibid.*). This usually meant that I asked a new question in response to Ethan, designed to make him think about the concepts he was struggling to master in a different way. In doing so, I believed I was taking the dynamic and individualised approach to supporting his advanced mathematical-development described by Belland, Kim, and Hannafin (2013) as motivational for learners engaged in problem-solving activities. This interpretation rested on the incorrect assumption that Ethan knew, instinctively or otherwise, the type of scaffolding that would best support him to transcend his ZPD in that moment. In Ethan’s case, Puntambekar's (2022) idea that learners tend towards the scaffolding strategy *being offered by the MKO* which best supports their development in that moment had been conflated with the idea learners know exactly what scaffolding would work best in all scenarios independently, and hence can engineer it when it is *not* offered by an MKO. It did transpire that Ethan was aware of the type of support he needed at later times in the study, analysed below (5.4.3, 5.4.4). Nonetheless, this analysis of findings relating to Diary Phase One suggests that nothing should be assumed about a gifted mathematician’s intention when reacting to a scaffolding technique during the early stages of advanced mathematical-development. This becomes a guiding principle in the developed pedagogical model (6.3).

5.4.3 Subtheme: Relating Through Scaffolding Strategies During Diary Phase Two

Confur wrote of the role of growing independence during Diary Phase Two. Most of his diary entries pertained to this in some way, and hence it became the single overarching individual emergent-theme for this phase in his advanced mathematical-development (4.2.3). His ZPD transcendence between the two diary phases was therefore significant. Analysis of his earlier views concluded that Confur sometimes found the most-intense scaffolding strategies (3.9.2) insufficient to support him to transcend his ZPD during the initial stages of advanced mathematical-development (5.4.2). By Diary Phase Two, he was describing experiences of independent success with mathematical problems, often seemingly not requiring any social interactions to be successful. Hence, his ZPD had transcended; tasks analogous to those which were beyond his skills even with intense support during Diary Phase One could now be described as his 'actual developmental level as determined by independent problem solving' (Vygotsky, 1978, p. 86) and what he was "capable of achieving unaided" (2.6).

Confur's descriptions of total independent success during Diary Phase Two meant he sounded reminiscent of the unchallenged gifted learner described by Barnett (2019); no longer feeling challenged by the problems he was working on, Confur was working beneath his ZPD. Hence, Özdemir and Isiksal Bostan's (2021b) argument would be that Confur needed even more challenging tasks at this stage of his advanced mathematical-development, in order to transcend his ZPD even further. This initially suggested that the problems selected for use during Diary Phase Two might not have been vicinal to Confur's ZPD. However, this interpretation is not entirely consistent with the intense sense of satisfaction Confur derived from his experiences of independent success. He felt satisfaction in successfully solving the problems. This demonstrates that he was, to some extent, aware that it was difficult in some way and hence he perceived it as a notable achievement (Petry, 2019). His earlier experiences of being temporarily fazed by similar problems in the way Starja, Nikolova, and Shyti (2019) described therefore facilitated positive perceptions of advanced mathematical-development from being able to do so independently at later times. This in turn highlighted the existence of opportunities for future success which motivated him (Czarnocha & Baker, 2021).

Confur was also able to choose the problems he worked on during Diary Phase Two (3.9.1). Like all participants, Confur's background of prior mathematical learning being beneath his ZPD is what motivated him to join the study and pursue advanced-mathematical development in general. It is therefore implausible he would derive immense pride from solving a problem he perceived as beneath his ZPD (2.6). This suggests that the problem Confur was referencing actually *was* vicinal to his ZPD. To further support this inference, it should be acknowledged that Confur made explicit reference to solving the problems independently given *'a bit of thought'*. He reflected on the need to think deeply, which demonstrates that the process of divining and implementing a correct method was not as simple as responding to a typical A-Level problem (OCR, 2024a, 2024b). My perspective on this is that Confur could confidently respond to the stimuli in an A-Level question, immediately knowing how to proceed and going on to concisely write out a solution without seeming to think too deeply, working entirely beneath his ZPD. His diary entry was therefore describing the impact of successfully navigating a process actually vicinal to his ZPD. He carried a level of uncertainty throughout the problem-solving process, which he overcame through sustained independent thought. Hence, Confur succeeded in an analogous process to that which Sriraman (2021) found was typical of professional mathematicians using uncertainty to fuel creative solutions to research problems. Although Confur did not write specifically of the scaffolding techniques in his diary entries, his description of how he derived satisfaction from problem-solving implied that the more-intense approach to selecting and implementing scaffolding strategies adopted at earlier times (3.9.2) would have eroded more of the independence he used to fuel his ZPD transcendence. Hence, utilising less-intense scaffolding techniques better supported him to transcend his ZPD during Diary Phase Two.

It was anticipated that the views of all participants would have evolved by Diary Phase Two in that they were responding to the less-intense approach to scaffolding (3.9.2). It is notable, therefore, that Derwyn wrote specifically about his perceptions of how interactions with me affected him for the first-time during Diary Phase Two, having not done so earlier. He described a scaffolding technique as being *'good when it made [him] step back and think of the problem in another way'*. Siklos' (2019) view that considering multiple approaches to a given problem is one skill to be honed throughout advanced mathematical-development supports a literal interpretation of Derwyn's perspective. This diary entry could therefore be taken to indicate that, by Diary Phase Two, Derwyn had realised the need to cultivate multiple approaches and was seeking out opportunities to learn about several methods whenever possible in pursuit of developing that skill. However, he later clarified at interview that the key phrase in his earlier diary entry was *'step back and think'*. His perceptions of advanced mathematical-development therefore benefited more from scaffolding techniques which supported him to take his time to think through a problem, rather than leading him directly to the correct method. The intention behind the choice of scaffolding strategies, to preserve the role of a gifted mathematician's independence, supported Derwyn to do this. For example, I would tell him he was close to a fruitful method, or gently hint at the avenues to think about next to guide his thinking (NRICH, 2021). These brief exchanges constituted social interactions between Derwyn and me. Hence, when such an interaction resulted in him making progress, this could be interpreted as the effect of adult guidance under Vygotsky's (1978) definition of the ZPD. However, Derwyn's perception that he was still required to think independently despite support led to more-positive perceptions of advanced mathematical-development than when more-intense scaffolding was utilised. In other words, a direct application of the original definition (*ibid.*) would have meant Derwyn positioned the problem he was tackling as beyond his current ZPD. Contrastingly, he actually positioned it vicinal to his current ZPD; despite having needed some support, that support did not erode his opportunity for independent problem-solving, and hence his perception was that he was able to solve the problem without needing much help.

This suggests the emphasis on a gifted mathematician's "current perception" as the mediator of their ZPD (2.6) faithfully captures Derwyn's experience. Moreover, the word "unaided" is not flexible enough to encapsulate that Derwyn felt he had succeeded with a high degree of independence despite needing some assistance, precisely because he perceived that assistance as minimal in that it enabled him to think for himself. This suggests that the adapted definition of ZPD requires updating. Describing these activities as "largely unaided" instead would make the adapted definition of ZPD more accurate with respect to Derwyn's experience. However, earlier analysis (6.4.1.3) concluded that success achieved with total independence was most powerful for other gifted mathematicians. The phrasing "entirely or largely unaided" is therefore more appropriate, being flexible enough to capture a variety of levels of independence while emphasising that the greater the role of independence is perceived by a gifted mathematician, the more positively they also perceive their progress.

Ethan reflected directly on the positive impact of the less-intense scaffolding techniques during Diary Phase Two more often than any other participant. He stated that *'I think all the tips and tricks that Niall gives (…) help a lot'*, further explaining that he subsequently '*underst[ood] it a lot better and [was] able to complete [the problem] with a bit of guidance'*. There is a distinction between *'tips'* and *'tricks'*; each of which holds its own analytical implications for further illuminating Ethan's perceptions of the scaffolding techniques during Diary Phase Two. What Ethan called tips has thus far in this thesis been referred to as hints, in particular when situated within the developed hierarchy of scaffolding strategies (3.9.2). Wrightsmant, Swartz, and Warshauer (2023) described hints as useful for scaffolding mathematical problem-solving. Their (*ibid.*) perspective was that hints should help a mathematician progress past the point at which they are stuck with a problem, but not excise what they refer to as productive struggle. This approach is intended to sustain the cognitive demand on the learner. Ethan felt he was able to use a tip to advance his understanding and subsequently complete the problem with *'a bit of guidance'*. This demonstrates that his perception of hints was consistent with the positive application of hints described above (*ibid.*).

Ethan's experience further justifies the use of hints as a scaffolding strategy with gifted mathematicians which facilitates progress whilst preserving the role of their own independence within their perception of their own success. However, the effectiveness of hints in this endeavour relies on the practitioner carefully considering when and how to give hints (Matsuda, Weng & Wall, 2020). During Diary Phase Two the approach was to use hints as the first scaffolding strategy (3.9.2). At this stage though, scaffolding strategies were only employed as an intervention when a participant was having difficulty making progress with a problem they had already considered in depth. Although hints were the baseline strategy, this did not therefore mean their application with Ethan entirely failed to utilise Matsuda, Weng, and Wall's (*ibid.*) perspective, that hints best preserve cognitive demand when implemented reactively rather than proactively. Ethan did not always explicitly ask for a hint; hence, hints were at times given despite being unsolicited, which the existing literature (*ibid.*) would have described as proactive, and hence ineffective. However, this is not entirely consistent with Ethan's perception of hints. He also wrote that he would improve further if he started the problems independently and only asked me for help after taking time to think, and hence only get support from an MKO when he really needed it (4.4.3). Although Ethan specifically mentioned asking for help, he offered this as one means through which he could ensure he really did need a hint to make further progress. An assessment of Ethan's progress with a given problem was used to determine whether he needed a hint in a particular scenario. Ethan's perspective therefore suggests that hints do not always require soliciting by a gifted mathematician to be useful in facilitating progress through sustaining productive struggle. Instead, it would be more effective to judge whether a hint is warranted based upon an assessment of progress, even if not asked for explicitly. Provided the practitioner designs the hints in such a way that a gifted mathematician's thinking is guided without depriving them of the feeling they are carrying a significant share of the cognitive load, the resulting progress can be perceived by the gifted mathematician as having been achieved "largely unaided", further supporting the introduction of the word largely in the adapted definition of the ZPD. This becomes the foundation of this scaffolding strategy within the developed pedagogical model (6.3).

Having considered what Ethan meant by *'tips'*, it remains to analyse the influence of *'tricks'* that he referenced in the same diary entry. Siklos (2019) described approaches which simplify the algebraic steps in a given problem so that the solution can be created with greater ease. This is an interpretation of what Ethan meant by *'tricks'*. Kumar (2023) detailed many other mathematical tricks, which he conceptualised as shortcuts to be applied in niche scenarios, but that are not strictly necessary when solving a problem. His perspective is specific to the single guise of advanced mathematical-development concerned with preparing for mathematics competitions, which require a significant amount of challenging mathematics to be thought about and written in a short amount of time (UKMT, 2024). STEP and other admissions assessments (UoC, 2024a) also place a time limit on successfully producing solutions to difficult mathematical problems; hence Kumar's (2023) perspective did apply to Ethan, who was pursuing elite university admission as part of his advanced mathematical-development. Indeed, Ethan directly reflected on tricks which enabled him to complete *'the question a lot faster and easier'*. This demonstrates that hastening the speed with which he could solve the problems was one important motivator within his advanced mathematical-development.

Ethan was able to benefit by applying a trick as a shortcut to save time (Kumar, 2023). However, this does not entirely capture the usefulness he perceived in tricks when developing as a problem-solver more widely. He also wrote of tricks which enabled him to complete the problems more easily. This reflected Siklos' (2019) emphasis on ease over speed (of problem-solving) within advanced mathematical-development. His (*ibid.*) view was that the challenges are associated with two different aspects of the problem-solving process: that of designing a solution to a problem; and that of implementing the solution accurately. Successful implementation is required extensively during A-Level assessment, but a learner's ability to design new and novel methods is not assessed as part of that qualification (OCR, 2024a, 2024b). However, the successful pursuit of advanced mathematical-development requires a gifted mathematician to begin conceiving of their own methods to problems, in addition to becoming efficient at implementing more-complex algebra which renders the implementation of those methods a more-challenging enterprise in its own right.

Gifted mathematicians must rise to the challenges of both conceiving of and implementing methods to difficult problems. However, my experience of these respective challenges in advanced mathematical-development mirrors the view of Sîntămărian and Furdui (2021) that mathematicians often find acquiring new abstract skills more difficult than further honing their existing skills. Siklos' (2019) position on mathematical tricks builds upon this. Tricks help streamline the process of implementation; in doing so, more of the mathematician's cognitive load is preserved; this enables them to focus more of their thinking into designing methods to solve problems, which they perceive as more difficult. This is illuminating when interpreting Ethan's perspective in that it suggests how Ethan's perceptions of speed and ease were interrelated. Without considering this nuance, teaching Ethan tricks to support his expedient implementation could be interpreted as an example of direct modelling, especially as Ethan described them as tricks *'that Niall gives'*, which placed responsibility for ZPD transcendence with me as the MKO. As the most-intense scaffolding technique utilised in the study (3.9.2), an initial interpretation of Ethan's meaning could be used as evidence that Ethan was not benefiting from the less-intense approach to scaffolding adopted during Diary Phase Two. However, the above analysis instead suggests that Ethan found the modelling of tricks useful in that it facilitated his focus on developing his ability to conceive of methods to the problems (Sîntămărian & Furdui, 2021), which by this stage he was perceiving as more difficult, and hence more important to focus on, than method implementation (Siklos, 2019).

Ethan's focus on the relative challenges of method conception and implementation was also captured in a diary entry relating to a specific problem during Diary Phase

Two. He wrote that *'I messed up in the last step (…) [and] didn't end up with the correct answer'*. Ethan had anticipated the correct answer using a sophisticated argument that relied on a subtle trick; he was subsequently able to identify his mistake by noting that his written answer was inconsistent with his anticipated answer. His mistake was therefore related to implementing a method he had successfully designed independently. Mathematicians often use terms like 'small slip' or 'minor error' when describing a mistake made during implementation (Shen *et al.*, 2021). This suggests they perceive their overall attempt at the solution positively despite this type of error (*ibid.*). Through succeeding to a great enough extent, they are able to recognise that the method they conceived of was valid (*ibid.*). This appears to run contrary to how a mathematical mistake might be interpreted from the Vygotskian perspective. In particular, there is a common view that one can either be right or wrong when solving a mathematics problem (Jansen, 2023). A mistake could therefore be interpreted as an indicator that a given solution is not entirely right, and hence must be wrong. Therefore, mathematicians might take their mistakes to mean they are currently incapable of succeeding in a particular task, alone or with support, and hence perceive their ZPD under either definition (2.6; Vygotsky, 1978) to have been subceeded. This was not the case for Ethan. Although he had made a slip in implementing the algebra, the method he had conceived of was sound; he therefore benefitted from knowing he had not made mistakes during method conception, the aspect of problem-solving described as the most difficult (Maulyda *et al.*, 2020; Shinariko *et al.*, 2020). Perceiving the error as minor meant he felt no material detriment to his ZPD. Although never leveraged as an intentional scaffolding technique, my response to the mathematical errors I made when demonstrating or collaborating on the problems (Aziz & Hakim, 2024) during advanced mathematical-development influenced Ethan through a social process of observation and direct imitation (Vygotsky, 1978). Implementation mistakes on my part were not uncommon occurrences throughout the study. When they happened, I perceived them as a natural part of advanced mathematical-development. My response to them was therefore a form of modelling. In particular, it demonstrated how to respond with the attitude that effective mathematicians make such mistakes regularly, and hence that mistakes do not have to affect their perception of their own capability negatively (*ibid.*).

Alvidrez, Louie, and Tchoshanov's (2024) found that a student's mathematical mistake can be framed such that the individual perceives an emphasis on their own capability. This informed my approach to supporting participants to rectify their implementation errors. In this case, upon reviewing Ethan's solution and identifying the algebraic error, I asked him to review a few lines of algebra in particular, to narrow his focus without explicitly pointing out the error. Intensifying the scaffolding (3.9.2) by narrowing Ethan's focus therefore framed his mistake as one he was capable of spotting alone, whilst acknowledging that his request for help indicated that he wanted support to identify it more expediently. Ethan subsequently identified the error, corrected it, and completed the problem. This suggests that, when supporting with implementation errors, a more-intense scaffolding strategy (Thompson, 2023) might not be perceived by the individual to have eroded the role of their own independence to the same extent it could have, in relation to a method conception error. Hence, practitioners should carefully consider whether the development they intend to support relates to conception or implementation, and the gifted mathematician's current developmental level with respect to both aspects, when selecting a scaffolding strategy of the appropriate intensity during later stages of advanced mathematical-development. This subsequently informed the approach to scaffolding within the developed pedagogical model (6.3).

5.4.4 Subtheme: Incorrect Assumptions of the More Knowledgeable Other

When preparing for the BPhO and NSAA, Ethan began to develop his problem-solving skills in physics in addition to maths. The original definition of the ZPD would still have recognised me as an 'adult' able to provide 'guidance' as an MKO (Vygotsky, 1978, p. 86). However, the original intention of the ZPD was to theorise childhood development as socially constructed, where most adults would be considered developed members of the society and hence already in the position children were developing towards (Barrs & Richmond, 2024). This does not hold true when applied to developing specialist knowledge, and in this respect the adapted definition's description of the MKO as "another person who is highly skilled in the task" (2.6) is more applicable in the given context. In Ethan's case, I no longer fit the definition as it pertained to the physics skills he was developing. He subsequently located an alternative MKO, Kindred, from whom he sought support in this endeavour. Ethan subsequently mentioned Kindred several times in his diary entries and during his interview. He stated that he felt the support from Kindred[4] was more collaborative, and in the later stages of the study would approach me only when he felt it was necessary for the solutions to be revealed. Prior to this revelation, I believed I had an accurate ongoing sense of the type of support Ethan needed. The scaffolding techniques employed therefore seemed highly individualised (Kim, Belland & Axelrod, 2018). Moreover, the intent during Diary Phase Two was to reduce the intensity of the scaffolding strategies (3.9.2), and hence intentionally be the opposite of what Ethan described. Answers were purposefully withheld to nurture his independence by selecting other scaffolding techniques that allowed him greater opportunities to think. Ethan was reflecting on the differences he perceived when two different MKOs utilised the two most-intense strategies identified in the original hierarchy (3.9.2), during a later stage of advanced mathematical-development. This initially suggested that the scaffolding strategy utilised during Diary Phase Two was itself unsound, at least in its application to Ethan. Hence, this aspect of his perspective merits further analysis which follows.

One possible explanation for the seemingly diverging perceptions of the relationship between Ethan and me was that Ethan's statement *'If I want to know the best way to do it, I come to you'* had been misinterpreted. This quote was initially taken to mean that Ethan perceived me as somebody to come to for the answers to be immediately revealed (GMI, 2019). However, Vygotsky's theory rested on an underlying

[4] The idea that Ethan and Kindred developed together inspired Kindred's pseudonym. Kindred, as in kindred spirit, meaning connectedness.

assumption that what an individual can do under 'adult guidance' (Vygotsky, 1978, p. 86) to some extent depends on how that adult is supporting the individual (Barrs, 2022). The same assumption applies to the help of "another person highly skilled in the task" (2.6) within the adapted definition of the ZPD. Learners can transcend their ZPDs at different rates when working with the support of distinct MKOs. This is therefore a notion at the heart of both Vygotsky's (1978) original theory and the application of it to advanced mathematical-development (2.6). Given that Ethan made this statement when asked to directly compare his interactions with Kindred to his interactions with me, he might, therefore, have actually been trying to indicate that he perceived his ZPD to be transcending more-easily under my influence than Kindred's. This is also plausible in that my experience working with gifted learners during their further education is significantly more extensive than Kindred's. Ethan might therefore have been perceiving my greater confidence in how to approach supporting him (Thompson, 2023) through the relative ease he felt he was able to transcend his ZPD under my guidance compared with Kindred's.

While persuasive, the first explanation of the differing perceptions of the relationship between Ethan and me is still vulnerable to critique in that it remained rooted in an assumption. In particular, it had been assumed that I, as an experienced MKO (1.5), was best placed both to determine which scaffolding techniques would be most effective when supporting a gifted mathematician in a specific scenario, and to actually enact those techniques to best effect within that scenario. Ethan's views directly contradict both of these assumptions. Ethan decided which MKO to approach depending on the type of help he thought he needed. This demonstrated that Ethan, somebody other than me, could determine for himself which scaffolding techniques would be most helpful. In other instances, the scaffolding strategy was enacted by Kindred, who Ethan seemingly perceived as more effective than me at enacting the collaborative approach to scaffolding in these instances. This raised two nuances for further consideration, which are analysed in more detail in the subsequent paragraphs. Firstly, that the participants might sometimes have been able to engineer alternative social processes to effectively support their advanced mathematical-development. Secondly, that the differences between Kindred and me might have affected how Ethan perceived our respective capability to support him to transcend his ZPD in specific relation to the scenario in which he requested our support.

Xi and Lantolf's (2021) perspective was that individuals tend towards alternative social processes when perceiving the current scaffolding technique as ineffective. When applied to the views shared during Diary Phase One, it was concluded that, during the initial stages of advanced mathematical-development, a participant’s attempt to engineer a social process is indicative that the current scaffolding strategy is ineffective. However, the alternative social process the participant tended towards were no more effective at supporting their development at that time. When reapplied to Ethan’s views during Diary Phase Two, the same perspective (*ibid.*) instead indicates that the social processes Ethan tended towards *were* more effective at meeting his needs as he perceived them; hence, they better supported his ZPD transcendence. Vygotsky's (1978) wider sociocultural theory suggests that, as an individual develops into the wider society through the support of social influences, they hone their power to influence the society in return. Ethan’s perspective might therefore be understood as a demonstration he had developed not only as a mathematical problem-solver between the two diary phases, but also as an individual who could purposefully interact with his environment (Bakhurst, 2023) and, through doing so, effectively support his own advanced mathematical-development. This further builds into the metacognitive understanding of the challenge of advanced mathematical-development (5.2.1) in that Ethan was now deliberately thinking about how best to further hone his metacognitive problem-solving skills. The subject of his thinking had become his own metacognition. This suggests he had grasped that his ZPD could be transcended through honing his thinking in relation to a variety of metacognitive layers (Drigas & Mitsea, 2021). In particular, Ethan had developed the ability to support his own metacognitive development, a consideration which at earlier times had been the exclusive concern of his MKOs. Hence, in later stages of advanced mathematical-development, gifted mathematicians might know the type of support they would find most helpful. This becomes a guiding principle within the developed pedagogical model (6.3).

5.4.5 Subtheme: Relationships Between Confur, Derwyn, and Ethan

When reflecting on experiences interacting with Confur and Ethan, Derwyn wrote only of the benefits he perceived through sharing the experience of tackling advanced mathematical-problems with peers. Initially, he took the view that two heads would be better than one, and hence to maximise their collective chances of success it was best that they work together (4.3.2). Derwyn and Ethan subsequently developed a particularly close relationship. They consistently chose to work together, both during the sessions and in their own time, which suggested their relationship was beneficial to both of them.

The first perspective which seemed to suggest a less-positive view of the interactions between Derwyn and Ethan was recorded in a brief diary entry made by Ethan during Diary Phase Two. He wrote that *'There was no single interaction that made me feel motivated or confident today'*. This was despite regularly working with Derwyn during the session that he was referencing. It is a recognised phenomenon that gifted mathematicians can grow beyond the people they previously found helpful for supporting their development (DI, 2020). Ethan was no longer reflecting on the helpfulness of his interactions with Derwyn. This might therefore be interpreted as an indication that he had outgrown Derwyn's help. However, Ethan subsequently went on to describe other helpful interactions with Derwyn. This suggested that he still perceived some helpfulness in their relationship when the original diary entry was recorded. It therefore became apparent that the critical word in Ethan's statement was *'single'*. He later confirmed when interviewed that he was actually trying to communicate that his motivation was being boosted through a different type of interaction than it did originally, and that he shared these interactions most often with Derwyn.

When offered what he considered to be too much help at too early a stage, Ethan's perceptions of his ability to transcend his ZPD were affected negatively (4.4.2, 4.4.3). Opportunities to succeed independently within the scaffolding techniques were therefore of great importance to Ethan. This is well supported by Vygotsky's theory (Hedegaard, 1992; Vygotsky, 1978) and the adapted definition of ZPD for gifted mathematicians (2.6), as succeeding without help is the defined baseline for what is beneath a person's current ZPD under both conceptualisations. Hence, feeling able to achieve more with less aid accelerated Ethan's progress. Only after an independent opportunity to think did he describe external support as being helpful for his ZPD transcendence. Likewise, Derwyn wrote that *'help was good when it made me step back and think of the problem in another way'* (4.3.4). Both Derwyn and Ethan wrote of the usefulness of each other for getting just enough help to think about the problem effectively, and then going on to enjoy independent success as a consequence. The role of a gifted mathematician's perceived independence was a guiding principle in the design and application of scaffolding techniques to support advanced mathematical-development in the study (3.9.2). However, that both Derwyn and Ethan felt they got this type of help more-readily from their organic interactions with each other than their premeditated interactions with me (Abtahi, 2017) was unanticipated.

Vygotsky (1978) wrote of an MKO's altruistic intent to support another's development influencing the spirit of the social interactions between them. This is usually interpreted positively in that a teacher's desire to help their students learn is what ultimately guides social interactions to positive educational outcomes (Newman & Latifi, 2021). However, the experiences of Derwyn and Ethan suggested that their awareness I held this same intent negatively affected their perception of how their direct interactions with me influenced them. They perceived me acting in the traditional sense of the MKO (2.6) who was purposefully attempting to support ZPD transcendence (Kim, Belland & Axelrod, 2018). Contrastingly, their intentions towards each other were not motivated by the desire of one to purposefully support the other. If at any point one of them was acting as an MKO with respect to the other in the traditional sense, they were never perceived by the other in that way. This meant that, when they successfully supported one another's progress, they did not perceive this as only being possible because of the intervention of somebody more knowledgeable. Instead, they credited themselves for this progress, which boosted their confidence in their capability. Their perception of the status of the 'other' was therefore an essential factor when judging what was vicinal to their ZPDs. This suggested that Vygotsky's (1978) original description of MKOs as 'adults' or 'more-capable peers' is consistent with neither Derwyn nor Ethan's perception of supporting each other. Neither perceived the other as 'more-capable' than themself. This is what ultimately imbued the development borne of their interactions with the greatest benefit for their perceptions of advanced mathematical-development. However, the actual capability of the 'other' remained an important factor in facilitating this mutual development (*ibid.*). Both had a similar high-level of mathematical knowledge and skill. This was essential in that they were each on the precipice of realising enough of their potential to rise to the level of challenge they were tackling together. Were the content not vicinal to both of their ZPDs, their mutual assistance could have proven too slight to be effective. Hence, the phrasing of the MKO as somebody who is "highly skilled in the task" within the adapted definition of the ZPD (2.6) better accounts for the co-development of gifted mathematicians of similar ability, whilst permitting traditional MKOs to maintain their positive influence. This analysis therefore ultimately contradicts the original negative interpretation of Ethan's perspective, and instead further corroborates the significance of Derwyn and Ethan's relationship to both of them.

Both Derwyn and Ethan described competition within their relationship. Working through experiences of being stuck is necessary when a gifted mathematician is working vicinal to their ZPD during advanced mathematical-development (Siklos, 2019). They therefore dedicate most of their focus and energy grappling with problems which, in that moment, they feel stuck with (2.5.3). However, they often do so independently while in a space shared with other gifted mathematicians, but when not actively engaged in social interaction with each other. They therefore often make assumptions about their relative speeds of progress, judging themselves negatively as a consequence. This perspective from the literature (*ibid.*) further informs an interpretation of my professional experiences of competition in advanced mathematical-development, that competition generally influences gifted mathematicians negatively. Derwyn and Ethan's mutual mentions of competition between them were therefore initially interpreted by me as negative aspects of their relationship. This was further corroborated by perceptions of competition which arose through the pilot study (Thompson, 2023); competition was also described by Confur in a negative way (4.2.2). However, neither Derwyn nor Ethan described anything of a negative nature when reflecting on their experience of competitiveness between each other, which they both explicitly spoke of in resoundingly-positive terms (4.3.4, 4.4.4). Żuromski and Pacholik-Żuromska (2024) adopted a stance on competition between social beings from a neurocognitive perspective. The development of some social primates is driven by a sociocultural need to compete with each other for finite resources (*ibid*). However, human beings have evolved beyond that need, instead being aware of the role of cooperation and communication in ensuring the wider group thrives (*ibid.*). In doing so, each member of the society has the potential to develop beyond the level they might have reached as a competitive individual (*ibid.*). Shephard and Santhakumar (2024) held a similar view on competitive individualism in direct relation to contemporary educational contexts. Their (*ibid.*) position is that the effective pursuit of higher social purposes requires the individualism which results from competition to be set aside in favour of collectivism. This inspires one interpretation of how Derwyn and Ethan avoided the aforementioned traps that led other gifted mathematicians to judge themselves negatively in relation to peers. From their respective views, it is obvious that each had a deep-rooted respect for the other as a problem-solver. Hence, they were kinder to themselves when making judgements about the relative speed of transcendence of their ZPDs. Characterising the competition they felt between them as *'friendly'* and *'healthy'*, even when discussing it in hindsight, suggests that their overall impression was one that ultimately supported them both to transcend their ZPDs. This is not to say that they never compared themselves negatively, just that those occasions were not the defining moments of their working relationship. My initial negative interpretation of their mentions of competitiveness therefore appears to be false.

The study was not intended to explore the dynamics of a pair of gifted mathematicians specifically. However, Derwyn and Ethan's relationship naturally took on undertones similar to those investigated by Abtahi (2017). Her (*ibid.*) study researched the co-development of two similarly-able mathematics learners pursuing appropriately-challenging activities. She (*ibid.*) concluded that such a relationship enables both learners to extend their ZPD without the need of a traditional MKO. Each scaffolds and supports the other on an informal basis (*ibid.*). This amounts to improving what each can achieve with help, and hence extends both ZPDs (*ibid.*). Observing such a relationship develop and go on to be successful was a promising observation in the study. If such a phenomenon could be emulated, there would be opportunities to support gifted mathematicians that do not depend directly on resources, particularly teacher time, which are not typically available to mathematics practitioners in many FE institutions. Moreover, despite how it has been represented thus far, the initial impression of this phenomenon was not unanimously positive. The impact of the pairing might have extended beyond its positive affect on Derwyn and Ethan. It was clearly beneficial for them. However, the impact of the pair's closeness on Confur is as yet unclear. Gifted mathematicians often feel abandoned or ostracised during further education (1.2, 1.5, 2.5.2). This suggested Derwyn and Ethan's closeness could have marginalised Confur by making him feel in some way left out. If this did have a negative impact on Confur, then making recommendations on this basis could amount to alienating some gifted mathematicians who are not as naturally social in order that others might enjoy more success simply because of their nature. Such a recommendation would fall foul of the same social justice criticisms levied against the current status of gifted mathematicians during their further education (2.5), which it has been argued routinely sees them fail to reach their full potential (2.5.3). Hence, it was important to explore this issue in more detail. To that end, an analysis of how Confur was influenced by his peers' partnership follows.

The way Confur's perceptions were influenced by the presence of peers in his advanced mathematical-development was also evident in his diary entries at all stages of the study. However, where Ethan and Derwyn usually described them in a positive way, Confur reflected on peers' influences affecting his perceptions of advanced mathematical-development both positively and negatively (4.2.2). In particular, he referenced negative influences most often during Diary Phase One. For example, he wrote that *'The problems are (…) very tough. (…) I also asked [peers] for help but it didn't change much'*, and hence he became less *'motivated to continue with the questions without [further] help'*. Ineffective strategies for supporting mathematical development, like the *'help'* Confur described getting from his peers when he asked them for it, have already been demonstrated to elevate the risk of a student disengaging with their mathematical development (Chand *et al.*, 2021). This offers a direct interpretation of Confur's views in this regard in that he stated he felt less motivated after not succeeding despite help being provided. However, during Diary Phase One none of the participants would have fit the description of an MKO that could support a peer's development. All were grappling with their adjustment to advanced mathematical-development at this stage to some extent (5.2.2), and hence could not readily be considered a 'more-capable peer' (Vygotsky, 1978), nor somebody "who is highly skilled in the task" (2.6). This might have suggested that Confur was making a false judgement of his ZPD by conflating his peers, who could not have acted as MKOs under both definitions of the ZPD at this time, with effective MKOs who should have been able to facilitate his transcendence. However, his comment about needing *'further help'* is an indication that he ultimately realised the help he received from peers was not as effective as it needed to be for him to continue making progress. From that perspective, Confur was, to a great enough extent, aware that his development was not being adequately supported by peers. He was therefore able to protect his engagement with advanced mathematical-development despite Chand *et al.*'s (2021) intimation that such an experience often causes mathematical disengagement. This suggests that Confur was aware that a different type of help than his peers had initially offered was available from an alternative source and would better facilitate his ZPD transcendence. Nevertheless, Confur's lived experience of support from peers was not positive. Keerthirathne (2020) wrote of two influencing factors impacting the effectiveness of peer learning that are relevant to Confur: student readiness, and teacher influence. This offers two insights into Confur's experience. During the early stages of advanced mathematical-development, Confur might not have been sufficiently developed to engage effectively with peer support (*ibid.*). This suggests that peer support is less useful for facilitating progress during the initial stages of advanced mathematical-development. Confur's negative experience of peer support also happened despite an MKO, who did meet the description under both the original (Vygotsky, 1978) and adapted (2.6) definitions, being present to scaffold his development, but failing to do so. I as the teacher in this instance did not, therefore, influence the effectiveness of peer development positively (Keerthirathne, 2020). Hence, this represents one way in which the peer-supported co-transcendence of ZPDs, like that shared by Derwyn and Ethan, can potentially go wrong during advanced mathematical-development. When Confur did not feel progress was forthcoming after being helped by a peer, he perceived his ZPD to have momentarily subceeded. His perspective is therefore evidence against simply allowing gifted mathematicians to pursue advanced mathematical-development together without purposeful interaction with, or the oversight of, a person acting in the traditional role of an MKO. The developed pedagogical model (6.3) therefore underlines the role of the teacher in all stages of advanced mathematical-development.

Confur also had positive experiences of working alongside peers throughout the study. He described one notable example during Diary Phase Two. In this instance, the presence of peers was the reason Confur perceived this particular experience positively. He derived motivation from perceiving his mathematical knowledge to have transcended to a higher level than his peers, and indeed me (4.2.2). He tackled a problem in a substantively-different way to everybody else, later stating that *'I was proud of how I understood parts of the question in a different way than (…) it was explained.'*. Lin, Chen, and Cheung (2024) wrote of the role of peer comparison and its potential influences on an individual's perception of their own cognitive and academic development. This suggests that Confur's perceptions of his ZPD were benefitting from his favourable comparisons with peers (*ibid.*). It should be noted that the boost Confur experienced in this moment was not evident until his data were analysed. He benefitted from a positive lived experience despite nobody else knowing of his achievement. Despite pertaining to remote education, Negara *et al.*'s (2021) perspective is illuminative in this regard in that it relates to mathematicians developing in shared environments where direct interaction is more limited than it would be during in-person activities. Their (*ibid.*) view is that, although an individual's perceptions of mathematical ability are internalised, peers are one influence on self-perception. This therefore suggested that the presence of peers against whom Confur could judge his own ZPD (Lin, Chen & Cheung, 2024) was essential in his ultimately forming positive perceptions of his ability to transcend his ZPD. This positivity was therefore not driven by a need to compete with peers in the way Derwyn and Ethan used competition as the medium through which they collaborated. Instead, it arose simply because Confur experienced this in the presence of other gifted mathematicians, against whom he could form a positive judgement of himself. To distinguish the two described types of interaction with the social context of advanced mathematical-development the participants benefitted from within this analysis, they are assigned their own specific terms. An interaction within which everybody it pertains to is actively involved, mutually exerting direct influences on each other, is henceforth referred to in this thesis as an interplay to emphasise the active nature of all those involved. Similarly, the adjective parasocial is employed in this thesis to reference interactions like those described by Confur as one-sided in that they existed solely in his mind.

Confur went on to purposefully curate positive parasocial interactions with the other participants during Diary Phase Two. His diary entries at that stage often reflected on him choosing activities that the other participants were not currently working on in the problem-solving sessions. In doing so, he ensured that he perceived his own success as achieved without their direct support. Hence, through interacting parasocially, his perceptions of what was vicinal to his ZPD were boosted by his knowledge that he had distinguished himself positively from peers. Parasocial interactions were not interplays; nonetheless, despite their one-sided nature, Confur's development through parasocial interactions were still situated in a social context. In his seminal work, Vygotsky (1978) posed the question: 'What is the relationship between human beings and their environment, both physical and social?' (*ibid.*, p13). Interplays between individuals are undoubtedly within this question's scope and indeed are discussed at length by Vygotksy (*ibid.*). However, he (*ibid.*) made no assumption that interplays were the only possible influences within the social context. Instead, the basis of Vygotsky's (*ibid.*) argument is simply that development cannot be understood or take place in isolation from the surrounding society. Situating development in a social context does not, therefore, mean development is driven solely by interplays between individuals (*ibid.*). In Confur's case, simply knowing that other gifted mathematicians shared the experience was sufficient to enable the independent transcendence of his ZPD. Interplays with other gifted mathematicians were not the significant drivers of this development.

The question regarding the extent to which Derwyn and Ethan's close partnership had a subsequent effect on Confur arose analytically through a concern that Confur's ability to transcend his ZPD might have been harmed. However, Confur went on to use the distance between himself and the other participants to curate parasocial interactions with them, which subsequently bolstered his ZPD transcendence. While he did have interplays with Derwyn and Ethan that affected these perceptions negatively during Diary Phase One, he used these moments when seeking out positive experiences of peers at future times. In this way, long-lasting damage of this nature was not evident. In fact, he was able to utilise parasocial interactions to fuel his ZPD transcendence. To characterise these interactions as solely negative is therefore not justified. The developed pedagogical model (6.3) therefore accounts for supporting gifted mathematicians to at some point pursue development through social interaction with peers in the way which feels natural to the individual.

5.5 Summary

This chapter has further developed the phenomenological findings, interpreting them to provide a collective perception of advanced mathematical-development. Moreover, it has analysed these perceptions in light of the theoretical framework and identified literature to critically evaluate their implications for the subsequent developed pedagogical model (6.3) of support throughout advanced mathematical-development. The Conclusions and Recommendations chapter which follows will set out how the nuances of this analysis were subsequently synthesised to create the model.

6 Conclusions and Recommendations: Future Practice and Research with Gifted Mathematicians During Further Education

6.1 Introduction

The primary purpose of this chapter is to synthesise the outcomes of all avenues of critical evaluation by presenting a pedagogical model to support advanced mathematical-development (6.3). This will serve as an answer to the following research questions:

1. How do gifted mathematicians perceive their experiences of advanced mathematical-development throughout the further education phase?
2. What implications do gifted mathematicians' perceptions of advanced mathematical-development have for effective pedagogical approaches which support them through the challenges they associate with this experience?

The presentation of a pedagogical model might at first appear to fail to answer the first research question sufficiently. However, the critical analysis which informed the model's creation was induced by the detailed perceptions of advanced mathematical-development contributed by Confur, Derwyn, and Ethan. Each participant's perceptions have already been exposited in detail and summarised (4.2, 4.3, 4.4). The purpose of the current chapter is to frame the model developed from their perceptions as recommendations for future practice with other gifted mathematicians (6.3). The model is therefore presented with references to what the participant's perceived, but without explicitly identifying which participants' perspectives it resulted from. All references to what gifted mathematicians might experience throughout this chapter reflect an important perception of advanced mathematical-development actually held by a gifted mathematician during their further education. In this way, this chapter also answers the first research question.

The pursuit of the knowledge which led to the pedagogical model (6.3) included many other innovations. This chapter therefore begins by summarising these wider contributions to knowledge, initially highlighting those that were scholarly, theoretical, and methodological. The pedagogical model is then presented and explained in detail (6.3), enabling its limitations to be discussed and avenues of future research suggested for further refinement.

6.2 Contributions to Knowledge

6.2.1 Contributions to Scholarly Knowledge

When reviewing the existing literature, it became apparent that although there was significant overlap between the fields of Giftedness, Mathematics, and Further Education, very little occupied their triadic intersection (2.3). The literature review therefore served to bind together relevant perspectives within these fields by critically evaluating each piece of literature based on its applicability to gifted mathematicians during their further education specifically (2.3). The core debates from each field were synthesised (2.3, 2.4, 2.5) to explore their potential combined influences on gifted mathematicians during their further education. This enabled a vast array of literature to be evaluated based upon its applicability to educational issues affecting gifted mathematicians during their further education specifically. In doing so, the literature review established how the disparate perspectives interrelated and were synthesised to fully comprehend the existing scholarly knowledge within the niche and analyse its implications, that this and future research might go on to make apposite contributions.

The research was not just an extended piece of scholarship into my own practice, but explored the perceptions of a niche group of gifted mathematicians to which I also belong (1.5). I was therefore particularly intertwined with the project (3.3). Moreover, as a gifted mathematician my perceptions of advanced mathematical-development were a further answer to the first research question in their own right (6.1). However, this did not mean it was appropriate or rigorous to introduce my professional experience without careful consideration. To that end, an approach to including professional experience within the critical analysis was developed (3.14; Van Beveren, 2024; Delve & Limpaecher, 2023; Hardjanto, 2022; Wang & Hu, 2023). This approach synthesises reflexivity considerations into the analysis by explicitly stating the professional experiences that led to them. However, reflexive views are presented alongside relevant perspectives from the wider literature, facilitating a critical evaluation of any underlying assumptions. This transparently informs the subsequent interpretation of data, elevating the rigour of the analysis by demonstrating how professional experience has been applied alongside independent perspectives to create meaning from the data. Including reflexivity considerations in this way could benefit future research where the topic of investigation is the researcher’s own professional practice, or where research and researcher are otherwise particularly intertwined.

6.2.2 Contributions to Theoretical Knowledge

Although Vygotsky's (1978) theory of the ZPD was identified for use with gifted mathematicians during their further education (2.6), its application with this group relied on several refinements. Language was developed to consistently describe ZPD evolution, and where activities were located with respect to a gifted mathematician’s ZPD. Moreover, the notion of the ZPD was reconceptualised to account for the nuances of advanced mathematical-development identified throughout the research. These nuances include: the focus on a gifted mathematician’s current perception as the ZPD’s mediator; describing the MKO as somebody “highly skilled” to highlight that not every adult could act in this capacity (5.4.1; Darmayanti *et al.*, 2023), and to allow peers to be interpreted as MKOs in some scenarios (5.4.5; Abtahi, 2017); not positively framing what is beyond the ZPD, to capture that a task feeling unfeasible even with help in the moment is what creates particularly-positive feelings of ZPD transcendence at later times (2.6; Czarnocha & Baker, 2021; Starja, Nikolova & Shyti, 2019); and phrasing what is beneath the ZPD as what can be achieved “entirely or largely unaided” (5.4.3; Wrightsmant, Swartz & Warshauer, 2023). The latter is to highlight that, while succeeding with total independence is the ultimate goal, succeeding after receiving support through less-intense scaffolding strategies only caused a gifted mathematician's ZPD to subceed to a minimal extent, if at all (5.4.3; Matsuda, Weng & Wall, 2020).

6.2.3 Contributions to Methodological Knowledge

The intent of the phenomenological approach in the research was to uncover detailed perspectives of the nuances of advanced mathematical-development as experienced by individuals (3.4). However, there were two predominant challenges to overcome when pursuing a phenomenological approach with gifted mathematicians. Firstly, that gifted mathematicians required structured support to disclose their perceptions in detail (3.10.1). Secondly, that a gifted mathematician's negative lived experience often goes on to fuel a positive hindsight reflection, meaning that participants also needed opportunities to reflect on their experiences in addition to describing them in the moment (3.10.2). The first challenge was addressed through the development of digital diary-interview method, a contribution of potential use in other research investigating the perceptions of mathematicians. The digital-diaries were tailored for use (Bartlett & Milligan, 2020) by gifted mathematicians in the following ways. Developed prompts and checklists supported them to write about how their experiences were affecting both their cognitive and affective development (3.10.1; Janssens *et al.*, 2018). The digital format enabled them to respond in a variety of media, to enable them to respond organically (Spence, 2019) and, in particular, to include the mathematics in their data as the means of focusing their thoughts on how the current experience was influencing their development (3.10.1). Diary data pertained mostly to lived experiences; hence, triangulating diaries with interviews six months after the diary periods introduced the necessary element of hindsight reflection (3.10.2; Ajjawi *et al.*, 2024; Candela, 2019). Moreover, interviews served as opportunities for participants to clarify any ambiguities, in particular the multimedia data, and for a first interpretation of their diary entries to be presented. This enabled them to guide and hence validate the phenomenological lenses applied during the analysis (3.10.2; Birt *et al.*, 2016).

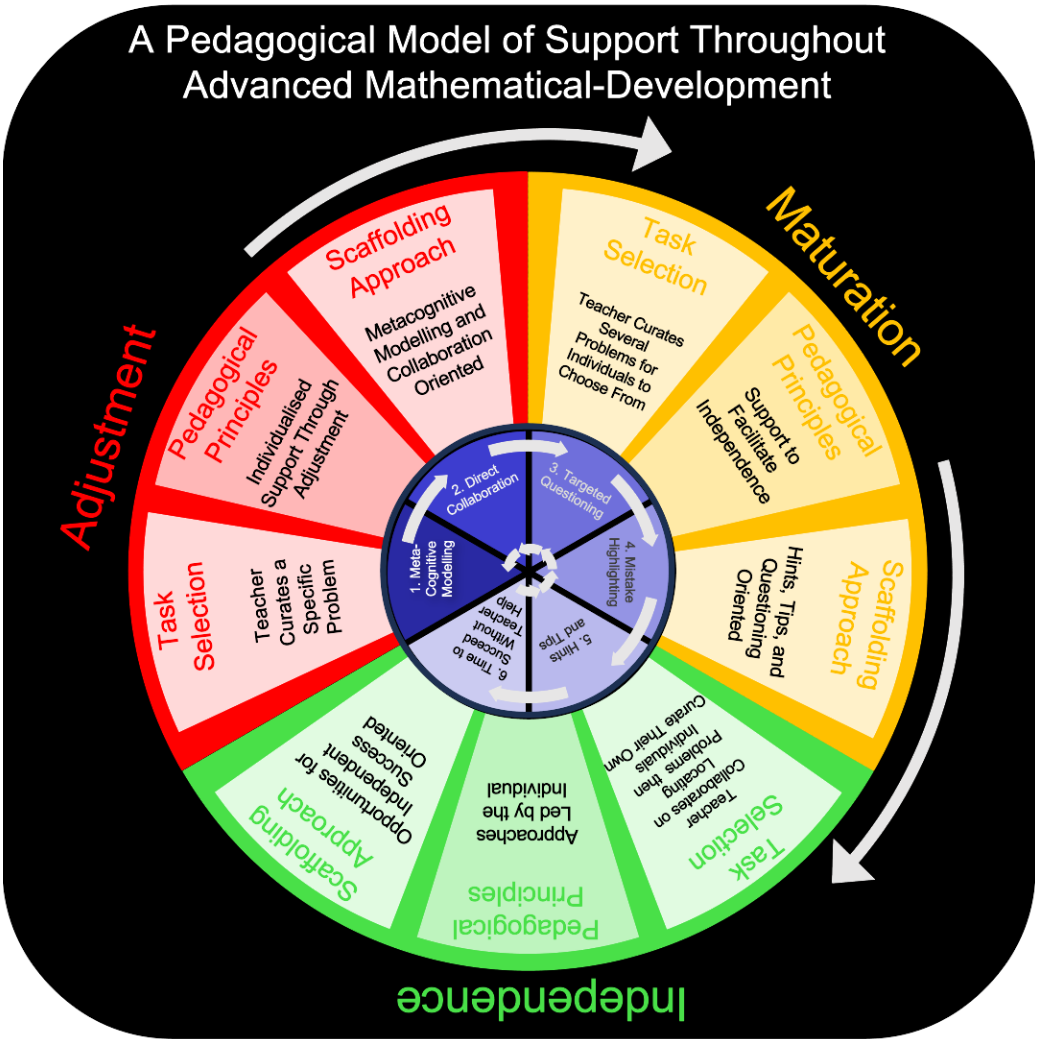
Diary-interview triangulation ultimately meant data arose in relation to distinct phases of advanced mathematical-development (3.6, 3.11). Interpretative chrono-phenomenological analysis (3.12) is therefore another methodological contribution of the research, developed as a novel analytical procedure to facilitate an evaluation of the evolution of each individual's perceptions of advanced mathematical-development. A traditional interpretative phenomenological analysis required refinement to facilitate this chronological evaluation (3.12; Latham, 2024; Smith, 2017). In particular, the developed analytical procedure not only involved separating each participant's data for individual analyses (Squires, 2023), but also further dissecting data into subgroups based the phase of advanced mathematical-development to which they pertained (3.12). Three separate phenomenological analyses were therefore undertaken for each participant in chronological order, resulting in three distinct sets of individual emergent themes (3.12, 4.1). Shared themes pertinent to all participants were then created in the traditional interpretative spirit (3.14, 5; UoA, 2024). The developed analytical procedure could prove useful in many social research projects where a detailed picture of an individual's evolving experiences is to be evaluated.

6.3 Recommendations for Practice: A Pedagogical Model of Support Throughout Advanced Mathematical-Development

6.3.1 Introduction

The study answered both research questions (6.1) through uncovering detailed perceptions of advanced mathematical-development (4.2, 4.3, 4.4) and critically evaluating these perceptions to identify the support needs of gifted mathematicians. In addition to describing effective practices for the valid identification of gifted mathematicians in a typical classroom during their further education (2.4.2), this knowledge was also sought to address the identified gap for a pedagogical model (see Figure 6.1) to guide practitioners on effective ways to approach supporting their gifted mathematicians. Its presentation is analogous to that of diagrams in solutions to mathematical problems. Specifically, the visual is depicted initially with just a brief summary of important details regarding how it was developed and its intended use by practitioners. Each subsection which follows then details how a specific aspect of the model was informed by the relevant outcomes of the preceding critical analysis (5).

Figure 6.1: A Visual of Pedagogical Principles and Scaffolding Strategies



The research identified that a gifted mathematician might have a period of adjustment (5.2) to work through before they are able to consistently perceive the benefits of feeling discomfort during mathematical challenge (5.3). After sufficient practice, they master this skill and are subsequently able to pursue advanced mathematical-development in a way that feels natural to them (5.4). The model is therefore intended to help practitioners to conceptualise the support needs of an *individual* gifted mathematician, by considering which of the three broad phases of advanced mathematical-development described above that the individual is currently aligned with. The rate at which a gifted mathematician progresses through these phases will be specific to the individual and vary over time. Moreover, it is possible that when an individual is between two consecutive phases, they might at times appear to regress and require intensified support.

Representing advanced mathematical-development as a circle emphasises that, although a gifted mathematician should make progress in general, progression through the phases is not necessarily linear and mono-directional. The traffic-light colour of each section represents the recommended intensity of the pedagogical approaches for a gifted mathematician within that phase, pertaining to both the choice of task and the means through which the individual is supported with it. The scaffolding strategies are placed at the centre of the model, as the means through which a practitioner can act to support the gifted mathematician. The depth of the shade of blue represents the intensity of the strategy (3.9.2). The strategies are also oriented such that those of appropriate intensity align with the phase they are most effective within (5.4.2, 5.4.3). However, the strategies can be cycled through in either direction, intensifying or abating in response to a practitioner's formative assessment of the individual in a given instance. Further details regarding the conceptualisation and use of the scaffolding strategies (6.3.2) and wider pedagogical principles underpinning each of the three identified phases (6.3.3, 6.3.4, 6.3.5) are given in the cross-referenced subsections which follow.

6.3.2 A Progression of Scaffolding Strategies

The notion of intensity of scaffolding techniques was referenced throughout the research (3.9.2). In particular, intensity refers to the extent a gifted mathematician perceives the role of their own independence to have been eroded by receiving support of that nature (Thompson, 2023). Perspectives on scaffolding from the literature (3.9.2; GMI, 2019; Khong, Saito & Gillies, 2019; NRICH, 2021; Szabo *et al.*, 2020; Wrightsmant, Swartz & Warshauer, 2023) were further honed to form what was originally conceptualised as a hierarchy of five scaffolding strategies, increasing gradually in intensity. The notion that highlighting mistakes can be utilised as an effective form of less-intense scaffolding arose from the critical evaluation (5.4.3) and is now included explicitly as a sixth strategy. The hierarchy was originally developed to guide how to intensify scaffolding strategies to tailor them to the needs of an individual gifted mathematician, with each strategy building upon the last (3.9.2). However, the analysis suggested that while working towards a greater sense of independence was important for all the participants, there were still occasions when they found intense support useful during the later stages of advanced mathematical-development (5.4.3). The strategies were therefore ultimately reconceptualised as a progression rather than hierarchy, placed at the centre of the pedagogical model to be cycled through in either direction depending on a gifted mathematician's support needs in a given scenario (6.3.1).

1. Modelling problem-solving processes through examples,
2. Collaborating with learners on the problems,
3. Asking questions to guide students to possible methods,
4. Highlighting errors without stating the mistake explicitly,
5. Providing hints to offer subtle guidance to methods, and
6. Giving extended time to create opportunities to overcome obstacles unaided.

6.3.3 Overseeing Adjustment

A period of adjustment at the onset of advanced mathematical-development was identified (2.5.3; Akkaya, Dogan & Tosik, 2021; Mofield & Parker Peters, 2018, 2019), with participants experiencing difficulty working vicinal to their ZPDs to various extents (4.2.2, 4.3.2, 4.4.2). There is a need to curate positive experiences throughout the adjustment period (5.2.3), so that gifted mathematicians can see what they stand to achieve by persisting with the adjustment despite their discomfort (5.2.1; Halmo, Yamini & Stanton, 2024). This suggested that carefully choosing a mathematical problem based on the prerequisite knowledge already being held by gifted mathematicians is most beneficial (3.9.1). Moreover, it suggested that intense scaffolding strategies would be best placed to support adjustment (3.9.2). However, the most-intense strategy is direct modelling (6.3.2). The analysis identified two pitfalls of modelling: a lack of individualisation when utilised as distributed scaffolding with several gifted mathematicians simultaneously (5.4.2; Puntambekar, 2022); and the need for a metacognitive approach to fully exposit the nuances of problem-solving in general, in addition to the specific methods for the problem under discussion (5.2.3; Avhustiuk, Pasichnyk & Kalamazh, 2018). While metacognitive modelling is likely to enable a distributed approach to scaffolding to be effective for a greater number of gifted mathematicians, it is still important that individual interactions with the teacher take place regularly (5.4.2; Kim, Belland & Axelrod, 2018). Hence, it is recommended that the metacognitive modelling of problems is broken down into smaller sections; after each small section, the teacher should ensure a meaningful interaction takes place with each individual gifted mathematician by collaborating with them on the relevant part of the problem. Moreover, tricks which help streamline method implementation should be modelled at this stage (5.4.3; Kumar, 2023). The aspect of problem-solving gifted mathematicians often find most challenging is the process of conceiving of methods to problems (Siklos, 2019). Through exposure to mathematics tricks (5.4.3), gifted mathematicians can allocate more of their cognitive load to this challenge, thereby focusing their efforts into the aspects of their advanced mathematical-development they need to prioritise (Sîntămărian & Furdui, 2021). Practitioners would also be well advised to model a positive response to their own mistakes from the onset of advanced mathematical-development (Aziz & Hakim, 2024). Gifted mathematicians can then begin to imitate this habit, thus acquiring the attitude that mistakes are learning opportunities from an early stage (Alvidrez, Louie & Tchoshanov, 2024).

6.3.4 Supporting Maturation

Once a gifted mathematician has lived through an experience which involved them persisting with metacognitive discomfort but subsequently succeeding, they might require further support to continue doing so consistently. Some are predisposed to this type of positivity (5.3.2). However, others might require support to establish the habit of processing the negative emotions they associate with challenge into positive feelings which fuel their motivation to continue (5.3.2; Greenspon, 2021; Snyder & Wormington, 2020). Structured opportunities to experience a negative emotion and begin processing it into a positive feeling to successfully develop a beneficial mindset (5.3.2; Lerman, 2019; Svendsen & Burner, 2023) should therefore be cultivated. At this stage, there is a fine balance to be struck between gifted mathematicians having sufficient support to succeed with the problem, while perceiving the role of their own independence as significant. A feeling of independence is what instils a belief they will be able to process emotions more positively in the future. The approach to problem choice therefore reflects this by the teacher curating several possible problems, and then allowing the gifted mathematicians to choose which to work on (6.3.1). This allows them to pursue problems they instinctively feel they have good opportunities to solve with a higher degree of independence. Likewise, the baseline scaffolding strategies should reduce in intensity (3.9.2). Giving a rough indication of where a mistake has been made in a solution is a particularly useful strategy for helping gifted mathematicians maintain their sense of independence when stuck at a particular point with a problem (5.4.3). They still have work to do to locate the error independently, but can do so more expediently. This approach also helps them see these mistakes as implementation errors they perceive as minor (Shinariko *et al.*, 2020), rather than a more significant method conception error they are trying to become more proficient at avoiding (Maulyda *et al.*, 2020). Targeted questioning can also be utilised, being particularly helpful when a gifted mathematician does not know where to start with a specific aspect of the problem. This scaffolding strategy guides their independent thinking into the topics which will prove fruitful when applied to a problem, but still requires them to think it through for themselves. Moreover, questions can be further tailored if it transpires that the individual requires additional support.

6.3.5 Facilitating Independence

When gifted mathematicians have become proficient at processing feelings of challenge associated with problems into the motivation to puzzle through them, they might also have developed their own sense of how to effectively approach their future advanced mathematical-development (5.4.4, 5.4.5). At this stage it is recommended that they are encouraged to locate possible problems for themselves, so they can pursue advanced mathematical-development with more independence (6.3.1). This might include collaborating on the same process the teacher utilised to identify the problems (3.9.1), so that gifted mathematicians learn how to locate and assess them for their own use. Support with the problems is now of greatest benefit when it helps the individual to think for themself (5.4.5; Siklos, 2019). This means that, in general, they find the least-intense scaffolding strategies more helpful. It is therefore recommended that a teacher utilises hints and tips to offer vague suggestions about which topics and ideas to consider (Wrightsmant, Swartz & Warshauer, 2023). However, a gifted mathematician taking a long time to conceive of a method should not be understood by the teacher as an indicator to provide immediate support (5.4.5). Hence, it is recommended that they ask the individual what sort of help they require (5.4.4). If the teacher judges that this type of help is potentially more-intense than the individual actually requires, they then have the option to cycle through the strategies to negotiate a middle ground which preserves as much independence as possible.

Gifted mathematicians will typically begin pursuing social interactions with each other to structure their own development (Bakhurst, 2023; Xi & Lantolf, 2021). There were two examples which arose in the research: regular interplays of healthy competition which drove two gifted mathematicians (5.4.5); and parasocial interactions where an individual only considered peers inside their own mind and structured their independent choices accordingly (5.4.5; Lin, Chen & Cheung, 2024). Both competition and parasocial interactions were initially believed to hinder advanced mathematical-development; the realisation both were actually sources of motivation only arose through extensive analysis (5.4.5). Nothing should therefore be assumed about how any type of social interaction pursued by gifted mathematicians is actually influencing their advanced mathematical-development (5.4.5). Teachers should maintain an ongoing conversation with each individual and regularly ask about their progress and the factors influencing it (5.4.5). This will highlight when the teacher can intervene with more-intense support (3.9.2), and when it is more effective to allow advanced mathematical-development to proceed organically (5.4.5; Abtahi, 2017).

6.4 Recommendations for Future Research: Limitations of the Pedagogical Model and Directions for Subsequent Investigation

6.4.1 Moderate Generalisability

Qualitative research is not considered generalisable (Hays & McKibben, 2021), and the small sample size in the study (3.8) restricted the extent to which its findings might be applied outside of its original domain even further. However, qualitative research can lead to moderate claims of wider applicability (Degtiar & Rose, 2023), provided the scenarios in which it would apply are clearly identified through their similarities to the experiences within the research (Johansson, 2021); caveats also need to be clearly expressed (*ibid.*). This allows moderate claims to be made in relation to the precise phenomenon under investigation (Levitt, 2021). The conceptualisation of a quality of mathematical giftedness (2.4.2) led to two key characteristics participants required to take part in the research: high performance in A-Level assessment, and the tendency to seek out all opportunities to develop mathematical skills (3.8). Although the specialist mathematics school setting in which the research took place was a niche in the FE sector, all participants were learners aged 16-19. Moderate claims about applicability to other gifted mathematicians with these characteristics in other FE institutions are therefore justified. This was further strengthened by the introduction of my professional experience within the analysis (3.14). My experiences of personally pursuing advanced mathematical-development and supporting other gifted mathematicians in various FE institutions since 2010 (1.5) were considered when interpreting the participants' perceptions (3.14). In this way, their individual perceptions were further connected with those of gifted mathematicians during their further education more widely. However, this also embedded my professional experience as a factor within the findings, making the pedagogical model that resulted from the research (6.3) particularly applicable to my own practice, the niche of specialist mathematics schools, and the specific gifted mathematicians who participated. Future research should therefore explore the model's robustness when applied by other practitioners, with other gifted mathematicians, and in a contrasting FE setting such as a sixth form college which is not a specialist mathematics school (1.5).

6.4.2 Sample Characteristics

Further similarities which facilitate moderate generalisability relate to the characteristics of the participants (Levitt, 2021). It was therefore encouraging that the sample reflected aspects such as a full range of FE ages (16-19) and included participants from three contrasting educational backgrounds (3.8). There were, however, some notable characteristics missing within it. In particular, there were no female or neurodivergent participants. The perspectives presented as findings are therefore both male and neurotypical in nature, and might not apply to gifted mathematicians without these characteristics. To investigate the robustness of the developed pedagogical model (6.3) and further refine it, future research would benefit from evaluating the model when applied with female and neurodivergent gifted mathematicians.

6.4.3 Constructionist Theories in the Theoretical Framework

The refinement of Vygotsky's theory as a theoretical framework (2.6) was one significant contribution to knowledge resulting from the study (6.2.2). However, the application of this theory to the findings ultimately required its underlying notions to be reinterpreted (5.4.5). Notably, peers began acting as MKOs (Abtahi, 2017). This was incorporated into the Vygotskian perspective through updating the description of MKO within the definition of the ZPD (2.6, 6.2.2). Once participants who experienced a period of adjustment had overcome its associated challenges, their relationships developed organically; each interacted with the others in ways they perceived supported their advanced mathematical-development effectively (5.4.5). This suggests that social constructi*onist* theories might facilitate further understanding. Such theories task groups of learners with using what they already know collectively to acquire new knowledge (Harris, 2022). Learners would be conceptualised as mutual mediators of learning within a group, and so drive learning for themselves and each other (Nickerson, 2024), rather than assigning to a small number of them the status of MKO who scaffolds the development of others on occasion. Hence, a theoretical framework which incorporates social constructionist elements might then be utilised to analyse the stages of advanced mathematical-development when the gifted mathematicians have become more intertwined socially in greater detail. In doing so, the pedagogical model (6.3) can be further honed by making more-specific recommendations for practice in relation to supporting gifted mathematicians to pursue advanced mathematical-development organically.

6.5 Reflections on the Doctoral Journey

This doctoral project began as I commenced my first appointment as a full-time teacher (1.5). My passion and interest for supporting gifted mathematicians during their further education phase has been a consistent feature of my professional motivation since before that time (1.5). However, the findings of my doctoral project have resulted in significant changes to my pedagogical philosophy and practice. I still take swift action when I observe a gifted mathematician struggling during advanced mathematical-development. At one time I would take this to be an indication I should intervene with support around the specific methods to be employed. However, the outcomes of my research have led to a re-conceptualisation of this experience, its significance for gifted mathematicians, and a practitioner's role in supporting them to perceive it developmentally. Through a rigorous investigation, I found that discomfort during appropriate challenge is not a feeling gifted mathematicians need to be rescued from. Rather, they require sufficient encouragement to persist with any metacognitive discomfort, that they might go on to rise to the challenge and, in doing so, develop as mathematical problem-solvers. Forming this critical distinction between learning to solve problems and honing problem-solving skills has been fundamental in the development of a pedagogical model (6.3) for supporting advanced mathematical-development. By taking a more-reserved approach to intervening, a gifted mathematician's opportunities to exercise and hone their problem-solving skills are preserved. This enables them to develop a sense of independent capability, which becomes their ongoing source of confidence as problem-solvers. In this way, I have shared the joy I have always felt in being challenged by mathematics with other gifted mathematicians, who also deserve to delight in this experience.

The knowledge which has emerged from my doctoral study has not only been transformative for many of the gifted mathematicians I have worked with and my teaching practice, but also for the field of study within which it is situated. As the first piece of empirical research at the triadic intersection of the fields of Giftedness, Mathematics, and Further Education (2.3), this doctoral project has established a new niche. What was a collection of disparate knowledge has been synthesised into a cohesive basis of scholarly work informing pedagogies for gifted mathematicians and highlighting apertures to be addressed by future research. It is my hope that my doctoral work will be just the first investigation of advanced mathematical-development, and that the practice and research which follows continues to improve the further education phase for gifted mathematicians.

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Appendix One: Researcher Personal and Professional Timeline

Table A1.1: Researcher Personal and Professional Timeline

September 2005 Age 11	My mathematical giftedness was identified upon starting secondary school. The teacher who identified it began to mentor me at an accelerated pace.
June 2007 Age 13	I undertook GCSE Mathematics and achieved an A* grade.
September 2007 Age 13	I began to study A-Level Mathematics part time at a nearby sixth form college, attending two hours per week alongside going to school full time. The school was reluctant to release me for any greater length of time and so I began on the single mathematics award. This was my first experience of the further education phase.
January 2008 Age 14	I undertook my first AS-Level examinations. 100% performance in both exams was the achievement which I subsequently used to negotiate additional hours at sixth form. I began studying Further Mathematics A-Level too.
June 2008 Age 14	I completed AS-Level examinations in Mathematics and Further Mathematics, achieving 2 A grades (the highest possible at the time). It was during this period I decided I wanted to pursue university applications alongside my peers in the autumn term.
September 2008 Age 14	I moved to full-time study at sixth form to ensure I could pursue the three A-Levels I needed to apply to university. I took A2 Mathematics and Further Mathematics, and self-taught both years of A-Level Statistics. I also followed courses in AS-Physics, GCSE Science, and GCSE English Language.
October 2008 Age 14	I applied to read mathematics at Magdalene College, University of Cambridge. I also began self-teaching STEP, the admissions assessment used by the university, at around this time.
January 2009 Age 14	I received my acceptance letter the day before my 15th birthday. I was made a conditional offer that included STEP in addition to A grades in all A-Level subjects.
June 2009 Age 15	I completed all my A-Level/GCSE exams, in addition to Advanced Extension Award Mathematics and STEP II and III. I secured the grades to meet my conditional offer.
October 2009 Age 15	I took up my place at Magdalene College. Throughout undergraduate study I taught in an FE institution regularly through volunteering at the sixth form college I attended at times of year I was not at university. This was my first experience working with other gifted mathematicians on advanced mathematical-development in FE, and I worked predominantly with other hopeful Oxford and Cambridge applicants.
October 2012 Age 18	Having graduated my BA, I enrolled on the MSc Applied Mathematics course at the University of Manchester.
January 2013 Age 19	I began working part-time at a tutoring agency alongside full-time study. I was placed in a variety of settings working predominantly with small groups of high-potential mathematicians of ages 14-19, on both GCSE and A-Level programmes.

October 2013 Age 19	I completed MSc study. At this point I decided to take a break in study and concentrate on tutoring gifted mathematicians, which had become more enjoyable than personal study.
January 2014 Age 20	Tutoring those during their further education on a full-time basis did not go as expected. Scaling up my teaching hours ultimately led to me being utilised predominantly with typically-developing mathematicians. I neither particularly enjoyed this, nor felt it was the best use of my knowledge and experience.
June 2014	Having made the difficult decision to step completely away from academia and teaching for a while, I took up a new role in a completely different industry.
May 2018	Desiring to return to mathematics teaching in an FE setting, I enrolled on a PGCE in Post Compulsory Education and Training (PCET).
September 2018	I began PGCE study and my teaching placement. The course was delivered by the sixth form college I attended as a student, and my placement was also within the institution.
December 2018	I began to feel disillusioned by the priorities for mathematics teaching in FE settings as they were being presented by some PGCE course tutors and teachers at my placement. I felt frowned upon by others for my desire to utilise my experiences to help other gifted mathematicians in the phase. Instead, I was encouraged to prioritise typically-developing mathematicians both in A-Level and GCSE. Doing more to support gifted mathematicians was framed as socially unjust, as accelerating those at the top 'widens the attainment gap'. It became clear that issues faced by gifted mathematicians during their further education were invisible or ignored by many in the sector.
January 2019	I began an action research PGCE module where I had freedom to choose the topic. This felt like the best positive opportunity to begin investigating the experiences of other gifted mathematicians and give them a platform to make their challenges during their further education known, and also to positively challenge the ingrained prejudice I was encountering.
May 2019	I graduated the PGCE and applied to the EdD programme.
September 2019	I took up my first full time teaching role in the independent sector. I predominantly taught A-Level Mathematics and Further Mathematics to high-ability (top set) classes. I also worked extensively with individual gifted mathematicians in the sixth form to support them with advanced mathematical-development.
October 2019	I began my EdD studies and project.
September 2020	I took up a new role at a specialist mathematics school, a type of sixth form college specialising in the mathematical sciences. This remains my current role, where I am head of the enrichment programme.
June 2021	I completed my doctoral pilot study.
June 2022	My proposal for the thesis stage of the EdD was accepted.
March 2023	My first article (Thompson, 2023) based on my pilot study was published in the Journal of Further and Higher Education.
December 2023	Doctoral research project final progress review undertaken and passed.

Appendix Two: Research Activity Timeline

Table A2.1: Research Activity Timeline

Month	Research Activity
September 2022	Began to consider which new Year 12 students might be suitable participants for my study.
October 2022	
November 2022	Presented the opportunity to participate to relevant students. Delivered all ethical messages regarding confidentiality and consent.
December 2022	Information packs were forwarded to participants desiring to participate and their parents/guardians. Consent and assent processes were completed before the school term ended, right before Christmas 2022.
January 2023	Diary Phase One took place between 23rd January and 6th February.
February 2023	
March 2023	Participants continued to pursue advanced mathematical-development at problem-solving sessions and in their own time. The participants were not recording diary entries at this time.
April 2023	
May 2023	
June 2023	Diary Phase Two took place between 12th and 26th June.
July 2023	Existing data from both Diary Phase One and Two was processed and then loaded into NVivo. Data familiarisation was undertaken on a participant-by-participant basis.
August 2023	
September 2023	Most participants were working on Oxford and Cambridge applications. Advanced problem-solving sessions took place regularly, but participants did not make diary entries at this time. Following the thorough period of data familiarisation, the interview schedules were tailored for each participant in readiness to take place in December 2023.
October 2023	
November 2023	
December 2023	Interviews with participants began to take place.
January 2024	The interview with the final participant took place, and all interviews were transcribed. Findings and analysis in relation to this phase were written up. First draft of full thesis completed 28th January.

Table A2.2: Summary of Confur's Participation in the Research

Brief Narrative of Educational Background	Confur was 16 at the beginning of the study and is white-Indian, having lived in the UK since birth. Confur attended an 11-16 state comprehensive, joining the mathematics school after an anticipated transition to a standalone sixth form college at 16. His aspiration was to study mathematics at university; he applied to the University of Oxford.
Information, Consent, and Assent Process	Information Documents Provided to Participant and Family: 1/12/2022 Consent Document Provided: 8/12/2022 Assent Process Concluded: 12/12/2022
Diary Phase One	Diary Phase One Commenced: 26/1/2023 Dates of Diary Entries: 26/1, 31/1, 2/2, 4/2
Diary Phase Two	Diary Phase Two Commenced: 27/6/2023 Dates of Diary Entries: 27/6, 29/6, 5/7, 6/7
Interview	Interview Date: 18/1/2024 Interview Duration: 57:32

Table A2.3: Summary of Derwyn's Participation in the Research

Brief Narrative of Educational Background	Derwyn was 17 at the beginning of the study and is white-British, having lived in the UK since birth. Derwyn attended an 11-18 state grammar school, joining the mathematics school after leaving his previous school earlier than anticipated at 16. His aspiration was to study natural sciences or physics at university; he applied to the University of Cambridge to read natural sciences.
Information, Consent, and Assent Process	Information Documents Provided to Participant and Family: 1/12/2022 Consent Document Provided: 8/12/2022 Assent Process Concluded: 12/12/2022
Diary Phase One	Diary Phase One Commenced: 26/1/2023 Dates of Diary Entries: 26/1, 31/1, 2/2, 4/2
Diary Phase Two	Diary Phase Two Commenced: 27/6/2023 Dates of Diary Entries: 27/6, 28/6, 4/7, 6/7
Interview	Interview Date: 18/1/2024 Interview Duration: 51:28

Table A2.4: Summary of Ethan's Participation in the Research

Brief Narrative of Educational Background	Ethan was eighteen at the beginning of the study and is white-British. Ethan lived in the middle east prior to joining the mathematics school, where he attended a private international school. His previous school did not observe the same educational phases as is typical in the UK for students of Ethan's age. In particular, the curriculum was broader, meaning that at sixteen Ethan continued to study many subjects rather than specialising in just three. Hence, despite being the age of a typical Year 13 student, Ethan joined the mathematics school as a Year 12 student. Ethan also lives independently of his parents, who remain in the middle east. Ethan's aspiration was to study natural sciences or physics at university; he applied to the University of Cambridge to read natural sciences.
Information, Consent, and Assent Process	Information Documents Provided to Participant and Family: 1/12/2022 Consent Document Provided: 15/12/2022 Consent (not assent) Process Concluded: 16/12/2022

Diary Phase One	Diary Phase One Commenced: 26/1/2023 Dates of Diary Entries: 26/1, 31/1, 2/2, 4/2
Diary Phase Two	Diary Phase Two Commenced: 27/6/2023 Dates of Diary Entries: 27/6, 28/7, 29/6, 4/7, 5/7
Interview	Interview Date: 18/1/2024 Interview Duration: 59:48

Appendix Three: Ethical Approval Documents

Figure A3.1: Ethical Approval (July 2022)

STAFFORDSHIRE UNIVERSITY

Institute of Education

ETHICAL APPROVAL FEEDBACK

Researcher name:	Niall Thompson
Title of Study:	SU_21_200 Gifted Mathematicians for Gifted Mathematicians: Effective Nurturing of Communities of Resilient and Motivated Gifted 16-19 Mathematicians to Support Higher Mathematical Self-Efficacy Throughout Further Education.
Status of approval:	Approved

Your project ***proposal has been approved*** by the Ethics Panel and you may commence the implementation phase of your study. You should note that any divergence from the approved procedures and research method will invalidate any insurance and liability cover from the University. You should, therefore, notify the Panel of any significant divergence from this approved proposal.

You should arrange to meet with your supervisor for support during the process of completing your study and writing your dissertation.

When your study is complete, please send the ethics committee an end of study report. A template can be found on the ethics BlackBoard site.

The Ethics Committee wish you well with your research.

Signed: **Date: 28th July 2022**

Dr. Sharon Inglis

Chair of the Institute of Education Ethics Panel

Figure A3.2: Ethical Approval (Amendment December 2022)

STAFFORDSHIRE UNIVERSITY

Institute of Education

ETHICAL APPROVAL FEEDBACK

Researcher name:	Niall Thompson
Title of Study:	SU_22_102 'Gifted Mathematicians for Gifted Mathematicians: Effective Nurturing of Communities of Resilient and Motivated Gifted 16-19 Mathematicians to Support Higher Mathematical Self-Efficacy Throughout Further Education.'
Status of approval:	Amendment Approved

Thank you for your correspondence requesting approval of a minor amendment to your Full Ethics Application SU_21_200.

Your amended application is approved. We wish you well with your research.

Action now needed:

Your amendment has now been approved by the Ethics Panel.

You should note that any divergence from the approved procedures and research method will invalidate any insurance and liability cover from the University. You should, therefore, notify the Panel in writing of any significant divergence from this approved proposal. This approval is only valid for as long as you are registered as a student at the University.

You should arrange to meet with your supervisor for support during the process of completing your study and writing your dissertation.

When your study is complete, please send the ethics committee an end of study report. A template can be found on the ethics BlackBoard site

Signed: **Date: 14.12.2022**

Dr. Sharon Inglis

Chair of the Institute of Education Ethics Panel

Appendix Four: Information, Consent, and Assent Documents

Figure A4.1: Letter of Invitation

Dear [NAME],

I hope this message finds you well. I am contacting you because I am currently planning a research project that [STUDENT NAME] has expressed an interest in taking part in. Although assessment and small-scale projects are undertaken in school all the time, this particular piece of research involves a greater level of participation. The research itself will only take place during school hours and on the school site, but also forms part of my doctoral studies and so is undertaken in partnership with Staffordshire University. In particular, this means that, should they ultimately take part, [STUDENT NAME] will need to contribute deeper information than is routinely collected in school, and that people outside of the school will have some access to any information they might contribute to the study for the purposes of advising the research overall. I therefore need to ensure that anybody who takes part is both well informed about what the study involves, and that their parent or guardian formally consents for them to participate before they can agree to participate in the research.

The attached information sheet and consent form offer you greater information about the study. The research itself will explore the ways in which new approaches to teaching and learning with gifted students can help them to become better and more motivated independent learners, and participants will be asked to create a journal over a two-week period to share their experience of taking part in the sessions. [STUDENT NAME] has already discussed this with me briefly and had an opportunity to read the information sheet, but I would encourage you to read it thoroughly yourself and discuss it further with them so that you all fully understand what participating will involve.

I must stress that [STUDENT NAME] will be more than welcome to attend their lessons and all other activities in school as normal even if they ultimately don't take part in the research, and please also be reassured that should you withhold consent or they change their mind at any stage, there will be no consequences at any point in the future. Please do not hesitate to contact me if there are aspects of this project you would like further information about, or need me to clarify, which will inform your decision.

You should also see the attached consent form, which if you decide to allow [STUDENT NAME] to participate I will need you to complete and return. Again, if there are any aspects of this you would like me to explain I will be happy to do so. If, however, you are willing to give your consent, then I would be grateful if you could complete and return the form, indicating yes to all the questions. At this point, I will talk with [STUDENT NAME] separately and ask them to confirm they also agree to take part independently.

I really appreciate the time you have taken to consider my research, and look forward to your response.

Best wishes,

Niall

Figure A4.2: Information Document (Parent/Guardian Version)

Title of study
Boosting motivation and developing autonomous learning confidence in naturally-gifted 16-19 mathematicians; a diary-interview study

Invitation Paragraph
I would like to invite your young adult to participate in this research project which forms part of my Doctor of Education (EdD) research. Before you decide whether you are happy for them to take part, it is important for you to understand why the research is being done and what their participation will involve. Please take the time to read the following information carefully and discuss it with others if you wish. Ask me if there is anything that is not clear or if you would like more information.

The purpose of the study
The purpose of this study is to better understand effective ways teachers can support the advanced learning of gifted mathematicians during their time during sixth form education. For many of these learners, a key factor in their success is having the motivation to work independently on challenging problems that they might initially need help and support with, overcoming any struggles they encounter positively and becoming more independent over time. This study will explore the effectiveness of social learning theories applied to these aspects of education, honing the underlying theories of learning specifically to enable the boosting of students' motivation to rise to the highest level of challenge, and to instil the self-confidence they need to explore these types of mathematical problems independently.

Why have I been sent this information?
My study requires three naturally-gifted students to participate. You have been sent this information because I believe [INSERT NAME] meets the necessary criteria to take part in the research. In particular, they are one of the school's high-attaining mathematicians, consistently attend the optional STEP sessions, and have mentioned in school that they might like to take part in the project when it has been mentioned informally. They have been given the same information that you have in this document, but as this project extends beyond the normal scope of classroom investigations in school it is important that you talk to them separately about the research and, if you are all happy to proceed, formally agree for them to take part. Should you give parental consent, I will then formally offer [INSERT NAME] the opportunity to participate.

What will happen if they take part?
Participants will continue to attend the optional STEP preparation (problem-solving) sessions and undertake any task set for homework as they would do normally. Those taking part in the educational research will be asked to provide data over two two-week periods, once in January 2023 and then again in May 2023. Each will be given an electronic diary in OneNote which they will use to record their progress throughout each two-week period in the study. They will write about anything they think is relevant to their learning, but will also have some questions to answer periodically. Participants may update their diary at any time, but opportunities will be planned in each of the sessions for them to do this too, meaning they will not need to invest a significant amount of their own time unless they choose to. If at any stage more information is needed or I need them to clarify something they have written, I might approach them separately to provide the details. Each two-week period will conclude with a 25-30-minute interview to discuss their experiences overall and further clarify their meaning if necessary. At the end of the study, participants will be given a copy of their diaries to use to guide their future learning.

Do they have to take part?
Participation is completely voluntary. You should only agree for [INSERT NAME] to take part if you are happy to give your consent. Withholding consent for any reason, or if [INSERT NAME] subsequently chooses not to take part, will not disadvantage them in any way. In particular, withdrawing from the research will not stop them from attending the sessions or taking an active part in them as part of their learning in school, or affect their ability to make good progress academically in any way. Once you have read the information sheet, please contact me if you have any questions that will help you decide whether to take part, and I will be happy to answer your questions in writing, or arrange a time to talk if necessary. If you decide to give your consent, I will collect the consent form and begin the separate process of formally asking [INSERT NAME] if they still want to take part. They will then be asked to complete a modified version of the consent form, known as an assent form, to formally agree to participate so that everybody has an opportunity to express their concerns. They will be given a copy of both the consent and assent forms to keep.

What are the possible risks of taking part?
There is no risk of physical harm associated with taking part in the study. However, the purpose of the study is to understand the ways in which altering classroom activities helps to stretch and challenge learners. In particular, there is an element of encouraging participants to think deeply about challenging problems independently before being helped by a teacher. While exposure to a heightened level of challenge is seen as a positive experience for many of our students, on occasion it can lead to feelings of anxiety associated with a novel and unfamiliar feeling of being intellectually challenged in a new way. I do not anticipate this risk is greater than in any other part of learning at ULMaS, however participants with an existing diagnosis of mental illness or a learning difference might be at increased risk. While you do not need to disclose any medical information to take part, I will, of course, be very happy to

discuss this in greater detail if you would like to. Should [INSERT NAME] ultimately decide to take part and experience any of these challenges, they will be able to access pastoral support by talking directly with me, approaching their form tutor, speaking with David Hemsley as the head of the pastoral programme. Time with a school counsellor can also be arranged if necessary.

What are the possible benefits of taking part?

As the main purpose of this research is to investigate the ways in which teachers can work more effectively with their gifted students, the main benefit of the study is that information the participants provide will be directly applicable to their lessons on an ongoing basis moving forward. Very little classroom research is undertaken with students of this age and ability, so the knowledge will prove extremely valuable to teachers across the 16-19 sector. Participants should also find it a worthwhile opportunity to think critically about their learning and independent study, assessing how effective it is at developing them towards their individual learning goals.

Data handling and confidentiality

Your data will be processed in accordance with the data protection law and will comply with the General Data Protection Regulation 2016 (GDPR), details of which are publicly available.

Data Protection (GDPR) Statement

Your data will be processed in accordance with the General Data Protection Regulation 2016 (GDPR).

The data controller for this project will be Staffordshire University. The university will process your personal data for the purpose of the research outlined above. The legal basis for processing your personal data for research purposes under the GDPR is a ‘task in the public interest’. You can provide your consent for the use of your personal data in this study by completing the consent form that has been provided to you.

You have the right to access information held about you. Your right of access can be exercised in accordance with the GDPR. You also have other rights including rights of correction, erasure, objection, and data portability. Questions, comments and requests about your personal data can also be sent to the Staffordshire University Data Protection Officer. If you wish to lodge a complaint with the Information Commissioner’s Office, please visit www.ico.org.uk.

What if I change my mind about my young adult taking part?

You are free to withdraw your young adult at any point of the study, without having to give a reason. Withdrawing from the study will not affect them in any way. [INSERT NAME] will be able to withdraw their data from the study up until four weeks following the final interview’s completion, after which withdrawal of their data will no longer be possible due to anonymisation. The precise date will be agreed as part of the consent gaining process. At this time, the data will have begun being analysed as part of a written report for my doctoral programme and potentially for future publication.

If you or they choose to withdraw from the study we will not retain any information they have provided us as a part of this study.

What will happen to the results of the study?

The predominant use of the results in this study will be to analyse any data provided for the purposes of writing a doctoral thesis. Participants will be anonymised and so not be identifiable in anything written about the study in the future. However, you should be aware that the anonymised data and the subsequent findings might be presented in published journal articles, at conferences, or as part of future professional development courses for teachers. Should you wish to be informed of the findings when the study reaches a conclusion, I will happily include you in my list of interested parties so you can read about what I found out in detail.

Who should I contact for further information?

If you have any questions or require more information about this study, please contact me using the following contact details: niall.thompson@liverpoolmathsschool.org

What if I have further questions, or if something goes wrong?

This research is being undertaken for the purpose of completing the final thesis of the Professional Doctorate in Education at Staffordshire University. If you have any concerns about this research, please feel free to contact my principal supervisor.

Principal Supervisor: Sandra Murray
Email: Sandra.Murray@staffs.ac.uk
Work: 01782 294315

If this study has harmed you in any way or if you wish to make a complaint about the conduct of the study you can contact the Programme Leader or the Chair of the Staffordshire University Ethics Committee for further advice and information:

Programme Leader: Gillian Forrester
Email: Gillian.Forrester@staffs.ac.uk
Work: 01782 294413

Chair of Ethics Committee: Tim Horne
Email: Tim.Horne@staffs.ac.uk
Work: 01782 295722

Thank you for reading this information sheet and for considering your young adult's participation in this research

Figure A4.3: Parent/Guardian Consent Form

UNIVERSITY OF LIVERPOOL MATHS SCHOOL
EDUCATION FOR 16-19 YEAR OLDS

STAFFORDSHIRE UNIVERSITY

RESEARCH PROJECT CONSENT FORM

Title of Project: Boosting motivation and developing autonomous learning confidence in naturally-gifted 16-19 mathematicians; a diary-interview study
Researcher: Niall Thompson

I have read and understand the information sheet.	Yes ☐	No ☐
I have been given the opportunity to ask questions, and I have had any questions answered satisfactorily.	Yes ☐	No ☐
I understand that my child's participation in this study is entirely voluntary and that I can withdraw them at any time without having to give an explanation without this in any way affecting my or their treatment now or in the future, in particular that withdrawing consent for their participation in the research will not stop them from taking part in any and all related sessions or lessons in school.	Yes ☐	No ☐
I consent that data collected could be used for publication in academic journals or be presented in forums such as conferences, seminars and workshops, or can be used for teaching purposes and understand that all data will be presented anonymously.	Yes ☐	No ☐
I agree that data will only be used for this project, although the data may also be audited for quality control purposes	Yes ☐	No ☐
I understand that all data will be stored safely on a password protected computer for 10 years before being destroyed	Yes ☐	No ☐
I understand that my child has the right to withdraw their data from the study, but that there will come a time when it is published or analysed at which point this won't be possible. I understand that they must withdraw their data before this happens on [AGREED DATE], and they will not have to give an explanation	Yes ☐	No ☐
I hereby give consent for my child to take part in this study, and for them to complete the assent form to confirm their desire to take part independently.	Yes ☐	No ☐

☐

Name Parent/Guardian (print)	Date	Signature
Name Researcher (print)	Date	Signature

Figure A4.4: Participant Assent Form

RESEARCH PROJECT ASSENT FORM

Title of Project: Boosting motivation and developing autonomous learning confidence in naturally-gifted 16-19 mathematicians; a diary-interview study

Researcher: Niall Thompson

I confirm that my parent/guardian has already formally consented to my participation in the study.	Yes ☐	No ☐
I have read and understand the information sheet.	Yes ☐	No ☐
I have been given the opportunity to ask questions, and I have had any questions answered satisfactorily.	Yes ☐	No ☐
I understand that my participation in this study is entirely voluntary and that either my parent/guardian or I can withdraw at any time without having to give an explanation without this in any way affecting my treatment now or in the future, and in particular that withdrawing my consent to participate in the research will not preclude me from participating in any and all related sessions or lessons in school.	Yes ☐	No ☐
I consent that data collected could be used for publication in academic journals or be presented in forums such as conferences, seminars and workshops, or can be used for teaching purposes and understand that all data will be presented anonymously.	Yes ☐	No ☐
I agree that data will only be used for this project, although the data may also be audited for quality control purposes	Yes ☐	No ☐
I understand that all data will be stored safely on a password protected computer for 10 years before being destroyed	Yes ☐	No ☐
I understand that I can withdraw my data from the project without having to give an explanation, but only have until [DATE] to do so before data is anonymised and analysed.	Yes ☐	No ☐
I hereby agree to take part in this study	Yes ☐	No ☐

Name Participant (print)	Date	Signature
Name Researcher (print)	Date	Signature

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Appendix Five: Mathematical Problems for Advanced Mathematical Development

Admissions assessment problems, for instance those from STEP (OCR, 2024a, 2024b), only require knowledge of A-Level Mathematics and Further Mathematics. However, it must be noted that even gifted mathematicians generally acquire this knowledge gradually throughout Year 12 and 13. Care is therefore necessary when selecting suitable problems for the purposes of advanced mathematical-development. I therefore evaluated potential questions by determining which topics a mathematician would need to be familiar with to successfully complete them. Cross referencing this with the school's scheme of work as it progressed enabled me to limit access to questions participants did not yet have sufficient prerequisite knowledge to tackle. This ensured that honing their problem-solving skills was the focus of their advanced mathematical-development, rather than learning a new topic from scratch. This process was aided greatly by my familiarity with STEP questions, built over decades through my personal education and subsequent teaching career (1.5). However, I developed two practices which would aid this process for a mathematics teacher without this familiarity in other FE settings. Firstly, STEP has historically been divided into three levels: STEP I (no longer offered) only required A-Level Mathematics topics (Glossary). STEP II additionally requires AS-Level Further Mathematics, and STEP III additionally requires A-Level Further Mathematics (OCR, 2024d). My participants were in Year 12 and follow a linear curriculum model where A-Level Mathematics is taught entirely before AS-Level Further Mathematics. This meant STEP I questions were, in general, more accessible in the earlier stages of the study. In schools operating a parallel curriculum model, where Year 12 students study the topics from AS-Level Mathematics and Further Mathematics at the same time before looking at the A-Level topics exclusively in Year 13, a different approach would be prudent. As Year 12s following a parallel curriculum model might have encountered some AS-Level Further Mathematics topics, some STEP II questions might be appropriate at an earlier stage. Likewise, if they have not yet encountered all of A-Level Mathematics topics, some STEP I questions might have been inaccessible. To give specific examples, there are many STEP I questions on integration methods which students would only meet in A-Level Mathematics and hence would be unsuitable. There are also many STEP II questions based upon the properties of roots of polynomial equations. Should a student have studied this topic, then many of these questions would be open and suitable for them to consider. Question choices also included examples from other admissions assessments. However, using STEP problems is one means of reducing the pool of potentially-appropriate questions to help teachers unfamiliar with them assess their suitability more efficiently. Secondly, the STEP database (stepdatabase.maths.org, 2024) has many STEP questions indexed by topic. Specific topics can be searched for in the database, significantly limiting the number of questions to be evaluated for use. This facility was particularly helpful, as new questions could be searched for as new topics were encountered in the A-Level scheme of work. This allowed a greater variety of question topics to be utilised as the study progressed. It also limited the volume of new questions to be evaluated. An example of a problem sheet developed for Diary Phase One is given overleaf.

Figure A5.1: Intervention Session Example Problem Sheet

5 (i) Write down the most general polynomial of degree 4 that leaves a remainder of 1 when divided by any of $x-1$, $x-2$, $x-3$ or $x-4$.

(ii) The polynomial $\mathrm{P}(x)$ has degree N, where $N \geqslant 1$, and satisfies

$$\mathrm{P}(1) = \mathrm{P}(2) = \cdots = \mathrm{P}(N) = 1\,.$$

Show that $\mathrm{P}(N+1) \neq 1$.

Given that $\mathrm{P}(N+1) = 2$, find $\mathrm{P}(N+r)$ where r is a positive integer. Find a positive integer r, independent of N, such that $\mathrm{P}(N+r) = N+r$.

(iii) The polynomial $\mathrm{S}(x)$ has degree 4. It has integer coefficients and the coefficient of x^4 is 1. It satisfies

$$\mathrm{S}(a) = \mathrm{S}(b) = \mathrm{S}(c) = \mathrm{S}(d) = 2001\,,$$

where a, b, c and d are distinct (not necessarily positive) integers.

(a) Show that there is no integer e such that $\mathrm{S}(e) = 2018$.

(b) Find the number of ways the (distinct) integers a, b, c and d can be chosen such that $\mathrm{S}(0) = 2017$ and $a < b < c < d$.

2 A curve has the equation

$$y^3 = x^3 + a^3 + b^3\,,$$

where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2 y - a^2 x = a^3 + b^3\,.$$

In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy

$$7x^3 - 3x^2 - 27x - 17 = 0\,.$$

Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3\,.$$

Appendix Six: Research Instruments

Figure A6.1 Research Diary Introductory Section

Information About This Research

Title of the Research:
Boosting motivation and developing autonomous learning confidence in naturally-gifted 16-19 mathematicians; a diary-interview pilot study.

Research Overview:
Thank you agreeing to participate in this research. I hope you are looking forward to taking part, and find the next two weeks to be a useful and enjoyable experience.

Over the next fortnight you will take part in four sessions focusing specifically on advanced problem solving. The sessions themselves will be designed with specific approaches in mind that are intended to challenge you to tackle difficult problems independently and grow in confidence with your approach over time. To assess the impact of these methods, I will be asking you to complete a diary of your progress, detailing how you feel about many aspects of the sessions and your independent studies. You may write in this diary in any format you choose and at any time you wish, but I will also be providing time for you to make entries at the beginning of each session, and to answer some specific questions at the end of each session. You can therefore decide for yourself the extent you want to engage with your diary over the next two weeks, what you want to write about, and how you choose to format it. You might also choose to include photographs or screenshots, or record voice memos or short videos if you feel that makes it easier to talk about your experiences.

After each session, I will give you access to a diary page in advance of the following session so that you can update your entries whenever you wish. There will be a list of prompts in each diary page to remind you what to talk about and how you might want to work with the diary. An image of what the diary will look like is included below. You may choose how to update your diary, but notepaper is provided so that you can make written annotations. If you decide you would prefer to type your answers, you are welcome to delete the notepaper and add text boxes. You can add images of your work or diagrams if you feel this is appropriate to what you are trying to say, or write out a few lines of algebra or even annotate written responses if you like. Do not feel bound by the space provided, you may copy and insert additional sheets of the notepaper if you need extra room, or delete any extra sheets you do not need. You may also choose to record voice notes or short videos instead of or in addition to your written answers and photographs, so please do this if you think it will help you to talk about the details of your experience.

Figure A6.2: Research Diary Sections for the End of Sessions

Questions at the End of the Session

Go To Question 1

Thinking specifically about today's session, please answer the following four questions. Feel free to draw diagrams, write a few lines of algebra or attach a photograph of the relevant parts of your work to talk about the maths itself if this feels right to you. If you do, please also write one or two short sentences to explain what you mean in more detail. I might ask you for more information at a later stage if I am not clear. You may also choose to record voice notes or short videos instead of or in addition to your written answers and photographs, so please do this is you think it will help you to talk about the details of your experience.

Question 1

Go To Question 2

What did you enjoy the most about today's session?

Question 2 **Go To Question 3**

Thinking specifically about any mathematics you undertook independently today, which aspects are you most proud of? You should think about:

- Why you picked these in particular, and
- What elements of the sessions so far, including today's session, have been particularly important in helping you to produce this mathematics independently today?

Question 3 **Go To Question 4**

Were there any areas of maths you felt stuck with today?
You should think about:

- What the challenges involved and why you felt this way,
- Any aspects of the session today which you found particularly helpful in beginning to overcome these challenges,
- The extent to which your interactions with the teacher helped you feel differently about the problems,
- The extent to which your interactions with the other students helped you feel differently about the problems, and
- How confident you feel about tackling this problem again more independently, even if you have not done so yet.

Question 4

Would you like to add anything else? If so, please detail below:

Figure A6.3: Research Diary Sections for Between Sessions

Prompts to Facilitate Diary Entries **Next Question**

Write about your experience of working on challenging mathematics problems since the last session. Any and all thoughts you have about your recent experiences when tackling these tasks are valuable to the research, so you should feel free to write about what you want and in as much detail as feels right to you. You may add details here at any time you choose, but I will also give you some focused time at the start of each session to update the diary regularly.

Here are some prompts to assists you in writing your diary.
• How prepared did you feel for advanced problem solving after the last session?
• How motivated have you been to complete the problems generally?
• How interesting have you found working on the problems?
• What mathematics you have done since last session are you most proud of, why, and how do you feel this has impacted your progress?
• Were there any areas you were stuck on or felt frustrated by? If so, what caused this, and how do you feel this impacted your progress?
• Were there any interactions with teachers or other students which impacted your confidence or motivation for tackling these advanced problems?

Please do not feel limited to these areas. You should write about anything you believe is relevant to your progress as an advanced problem solver. If you would like to, you might choose to write some lines of algebra or include mathematical diagrams, or insert photographs of your work, especially if this feels like a clearer way of talking about the maths itself. If you do so, please also write a short sentence or two which explain how it is relevant to what you are trying to say. I might ask you separately if I need more information.

You might want to annotate the note paper below electronically, or you might prefer to type your responses. You may delete the note paper and add in text boxes if so. You do not need to fill all of the lines, and equally should add in extra sheets if you need to.

Figure A6.4: Research Diary Checklist for the Beginning of Sessions

Checklist for the Beginning of the Session **Previous Question**

Please complete the checklist to state you have detailed an answer to each of the specific points mentioned in the previous question.

I have mentioned:

	Yes	No
How prepared I felt for advanced problem solving after the last session	☐	☐
How motivated I have been to complete the problems generally	☐	☐
How interesting I found working on the problems	☐	☐
The mathematics I did that I am most proud of and why	☐	☐
The areas I was stuck or felt frustrated and how this impacted my progress	☐	☐
Any ways that my confidence and motivation were impacted by interactions with teachers and other students.	☐	☐

If you have selected 'No' for any of the questions, please either add some more details to your original responses or make more comments about them below to fill in any gaps. Again, you may either use the notepaper below to annotate or add in text boxes if you would prefer to type, and write as much as you feel is right.

Figure A6.5: Representative Sample of Handwritten and Audio Data

Question 1

Go To Question 2

17 March 2021 13:19

UNIVERSITY OF LIVERPOOL MATHS SCHOOL EDUCATION FOR 16-19 YEAR OLDS

STAFFORDSHIRE UNIVERSITY

What did you enjoy the most about today's session?

I enjoyed that we were able to think carefully about the question before we had even had a go at it. I think that it allowed me to think carefully in a lot of different ways. This meant that when and I came to the step session, we could share our ideas which turned out to help us in the question. I also liked the fact that it was a small session (being only Niall, and I). I think that it was a good number of people, we could share our ideas in depth and easily bounce off of each other's ideas. I also liked the feeling when I was overthinking the problem, it was quite a nice and simple solution, but I was massively overthinking it so when Niall told us the answer it gave me that "ohhh ah ok!" feeling (I'm sorry I'm not sure how to explain it but it basically made me feel stupid in a good way). I loved the satisfying solution of the question and the fact that when I understood how to get to that solution easily, I had that "ahh" moment (a sudden moment of realisation).

Good stupid - the feeling when the solution was easy, but the thinking behind it is difficult, so when you find the solution or are told how to do the question, it is the feeling of "I am so stupid for not thinking of it", however it is a good feeling, because although it's slightly frustrating that you didn't know the answer, you know that you have gone through the process of thinking about it, even if you didn't get the right answer.

AUDIO

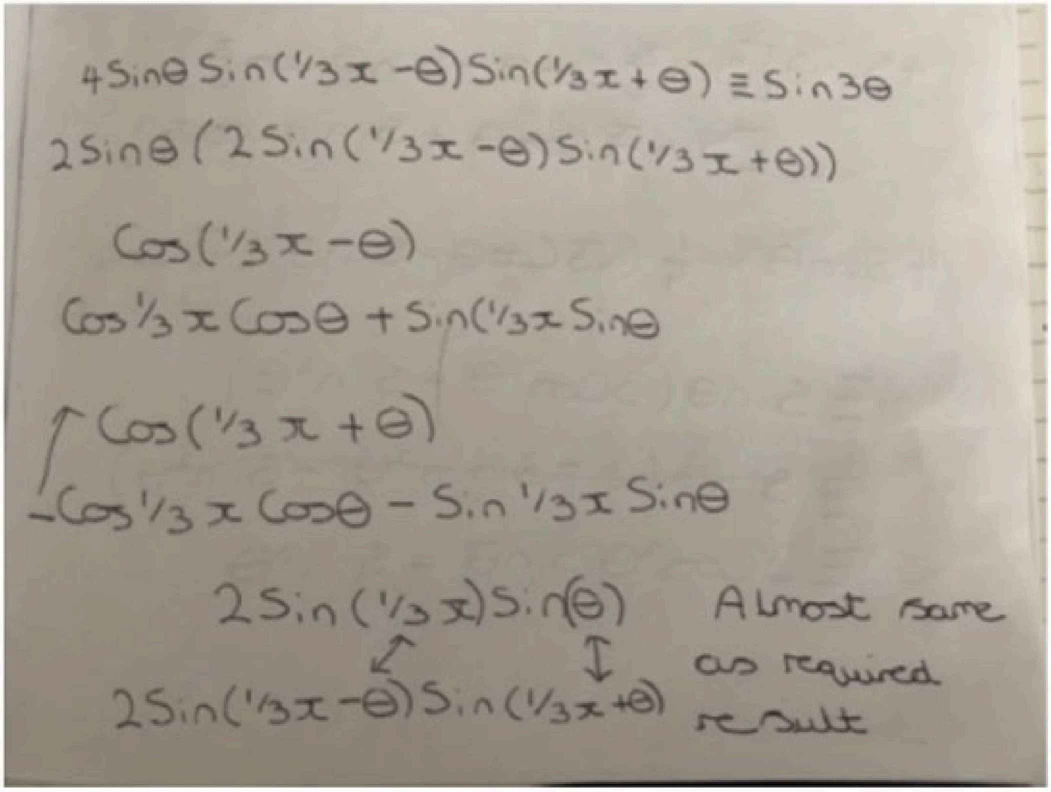
Figure A6.6: Representative Sample of Photograph and Typeset Data

*

Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,.$$

**



During today's session we looked at the 2011 STEP 1 Paper and tackled question 3 (*) in a small group. I enjoyed that in today's session I was able to demonstrate the understanding that I had done independently before the session. At the start of today's session Niall asked about what progress we had made with the question and I told him about the independent work I had done on the question. I then showed Niall on the board the simplification I had made (**) and explained it while doing so. I really enjoyed this part of the session as this is not something we normally do and by having to write the progress I had made out on the board it meant that I had to explain it as I was going along. This allowed me to not only display the maths I had done but also gave me the opportunity to justify to myself through explaining it. Whilst I admit it was problem not the best explanation, it felt good to try and explain it out loud rather than on paper. Furthermore I enjoyed the fact that after I had explained my way of thinking about the question Niall then pointed out that he had thought about it in a completely different way. I had used the trig identifies to simply the expression and break it down whereas Niall had split the expression into a difference of two squares (***) so that he could expand out and eliminate parts of the expression. On this occasion It was interesting to see a different way of coming to the answer. The reason I say on this occasion is because sometimes seeing lots of different ways of solving a question can become a little bit overwhelming and confusing trying to decipherer and understand each one. But because on this occasion the other method was mine I already understood mine way of doing it. The only method I needed to 'decipher' was Niall's so it didn't feel like I was randomly jumping from method to method which sometimes being shown different methods can do. Another thing I was pleased by in today's session was that at the end of the question it was one of the first time's I felt as though if I were to now attempt the question again I could possibly do it without any help. I don't normally feel this way with STEP questions and I think this was a very rare question that I simply had a good understanding of it but it was definitely helped by my own independent work, being a topic I enjoyed and Niall's thorough and well explained walk through today. Lastly, as always I enjoyed the structure of today's session again. Similar to last time it involved Niall working through the question and asking us for inputs along the way and then occasionally allowing us to justify why we did parts of the work this style of session is the best for me.

Figure A6.7: Interview Schedule Template

UNIVERSITY OF LIVERPOOL MATHS SCHOOL
EDUCATION FOR 16-19 YEAR OLDS

STAFFORDSHIRE UNIVERSITY

RESEARCH PROJECT INTERVIEW SCHEDULE

Title of Project: Boosting motivation and developing autonomous learning confidence in naturally-gifted 16-19 mathematicians; a diary-interview study.

I Opening (1-2 minutes)

- (Establish Rapport) [shake hands]
  Thank you for taking the time today to talk to me a bit more about your experience participating in the recent sessions.
- (Purpose)
  This interview will give you the chance to further elaborate your views and explore your experience of working closely with the diaries as part of the earlier phase of the research.
- (Plan for this Interview)
- The interview will be in two parts. Firstly, I will ask you to reflect on your experience taking part in the recent two-week trial. I will then ask you some questions about specific aspects of what you recorded in your diary, and ask you to confirm that my understanding is correct.
- (Time Line)
- The interview should take no longer than 25-30 minutes.
- Throughout the interview, I will ask you about your experience in the study overall, and specifically about the time we spent working together. I will refer to the times we spent working together as 'sessions' and the two-week periods overall as 'the trial periods' to help you think clearly about both of these aspects. If at any stage you are unsure what I am asking about, please do ask me to clarify.

II Overall Experience of Participating in the Trial (8-10 minutes)

Transition: Let me begin by asking about your experience taking part in the recent trial. I would like you to think about the experience overall, including all the sessions in addition to anything that you did independently. For example, you might want to talk about other STEP club sessions, any times you were solving problems outside of a lesson either independently or with another student, or any of your preparation for admissions assessments and interviews. You should feel free to refer to your diary throughout this interview if it helps you to answer the questions I ask, but you do not need to do this. It is perfectly fine to answer from memory.

1. Tell me about the times you have felt particularly motivated.
   (a) Do you feel the trial overall affected your motivation?
   (b) If so, what aspects of the sessions made this difference in particular, and how did they affect you?

2. Tell me about the mathematics you have produced that you are most proud of.
   (a) Do you feel the trial overall affected your ability to tackle the problems independently?
   (b) If so, what aspects of the sessions made this difference in particular, and how did they affect you?

3. In what ways have the other students taking part influenced how you felt? Are there any specific examples, either positive or negative?

4. What aspects overall do you feel have been most useful for your progress?

5. What aspects would you change to make them even more useful?

III Clarifying Diary Entries (8-10 minutes)

Transition: Next we will talk specifically about how I have interpreted some of the things you recorded in your diary. Again, you may refer to your diary as we go along. It is important you tell me about how you felt at the time, even if that has changed slightly in hindsight. Please also let me know if you feel differently now.

1. We have had to delay various parts of the research so that you and the other participants could quite rightly prioritise key points in the academic year. For example, we moved the second diary period to June to work around end of year exams. We then delayed this interview first so that you could focus on university admissions assessments and interviews, and more recently so avoid the mock exams. Can you tell me how this affect you?

2. Were there any things that happened outside of school that affected how you felt about becoming a problem-solver? This could be to do with friends and family, life events, or anything you think is relevant. If so, please tell me about them.

3. Other questions relevant to the individual participant.

V Closing

Thank you very much for participating in the trial. I hope you have found the experience enjoyable. I would like to remind you that the data in your diaries has already been analysed and cannot be withdrawn. Should you wish to withdraw the data you have given me today in this interview, you may do so at any time before 25th January, at which point I will have analysed it. If you have any questions about this study, you may ask me about it at any time. I will also provide the findings of the study after the data has been fully analysed and my thesis has been written.

Appendix Seven: Examples of Codes, Categories, and Themes

Table A7.1: Example of Chrono-Phenomenological Codes and Categories

Individual Emergent-Theme Category Code Subcode	**Individual Emergent-Theme Category Code Subcode**	**Individual Emergent-Theme Category Code Subcode**
Adjusting to Abstract Questions	Speed of Understanding	Standing Out from Others
Different Style of Problem	Positivity When Understanding	Help From Others
Didn't Know Where to Start	A Moment of Catching On	Asking People Around Me
Difference Between A-Level Questions	Answering Without Help	Support From Friends Did Not Help Much
First Step Given to Confidently Know What to Do	Enjoyed Seeing the Solution Just Appear from a Mess of Calculations	Messaged Some Friends for Help
Getting Used to the Style of Questions	Interesting When I Got It	Personal Uniqueness
How You Can See Solutions in Advance	Mostly Understood the Question	Differently than Everyone Else
Knowing Where to Go	Nice When I Understand What Was Happening More Easily	Think About It Visually
More Abstract Questions	Proud of Understanding	Think I Will Be Much Better
Nervous About the Next Question	Rewarding After Understanding	Understanding in a Different Way
Questions Are Very Different	Stuck on Understanding	

Individual Emergent-Theme Category Code Subcode	Individual Emergent-Theme Category Code Subcode	Individual Emergent-Theme Category Code Subcode
Feeling Stuck	Success After Understanding	
Frustrating When Stuck	Understanding More Easily	
Help From Friends Did Not Help Much	Speed of Cognition	
Help When I Was Stuck	Quick Thinking Affects Motivation	
Overriding Feelings	Demoralising when Speed is Fast	
Consistent Struggle	Explaining Slowly	
Demoralising when Speed is Fast	Explanations Too Fast	
It Feels Tough Often	Falling Behind	
Misunderstanding Is Stressful	Proud of the Speed	
Think I Will Be Much Better	Speed of Explanations Confusing	
Wasn't Very Motivated to Continue Without Help	Stressful When Speed is Fast	
	Slow Thinking is Productive	
	Explaining Slowly	
	Reading Through Slowly	

Table A7.2: Example of Interpretative Codes and Categories

Shared Theme Category Code Subcode	Shared Theme Category Code Subcode	Shared Theme Category Code Subcode
Feelings of Frustration, Motivation, and Independence	Adjusting to Mathematical Problem-Solving	Relationships with Other Gifted Mathematicians
Frustration	D Concentration	D Asking for More Questions
D Frustration with Lack of Progress	Concentrate Even When I Thought I Knew it Already	Asked for More Harder Mechanics Questions
Frustrated I Could No Longer Complete the Questions	Concentrated for a Short Period of Time	Challenging Mechanics Question
Frustrated When I Had Not Made Progress After 10 Minutes	Engaged and Happy with the Task	Mechanics Question Challenged Me
Frustrated When Knowledge Came to an End	Going Forward I Tried to Fully Concentrate	D Help
Frustration Makes Me Feel Very Excited for Further Maths Session	I Never Got Stuck When I Tried to Concentrate	Good Help Made Me Step Back from the Problem
D Getting Stuck	Stuck when I Hadn't Fully Concentrated	Help Led to Correct Answer
Doubt I Could Finish the Question	D Improving Ability	Help Made Me Look at the Pattern
I Struggled with This Question	Eventually Be Able to Tackle Any Question	Help Was Good
Still Think I Can Improve Quite a Lot	Hopefully Be Able to Tackle Any Question	Nice when Help Helped
Stuck Trying to Express Intent	I Would Like to Improve Further	E A Little Help from Niall

Shared Theme Category Code Subcode	Shared Theme Category Code Subcode	Shared Theme Category Code Subcode
Stuck When Trying Too Hard	Improving my Mathematical Ability	All Big and Small Tips Are Important
E Feeling Stuck	It Could Take More Practice	Complete it with a Bit of Guidance
Attempted the Second Part	D Mastering Questions	Enjoyed Question Without Much Help
Didn't Get Correct Answer	Full Understanding	Figure Out with Only a Little Help from Niall
Getting Stuck on Last Part Happens Often	Biggest Encouraging Factor	Niall Pointing Out Arithmetic Mistake
Got Question Wrong	Fulfilling to Complete a Whole Question	Niall's Tips and Tricks Help a Lot
Got Stuck on Second Part	Get What Was Asked For	Niall's Tips Showed a Faster Way
Managed to Get There in the End	Getting the Answer	No Interaction Made Me Motivated or Confident
Messed Up the Last Step	Satisfying Aspect of Finishing a Question	Tips to Speed Things Up
Overcome Frustration with Slower Thinking	Understand the Method	E Seeing How It's Done
E Frustration Becomes Enjoyable	Understanding All the Steps	Fun Seeing How Niall Did It
Figuring out the Proof by Contradiction was Frustrating	Understanding it Fully	Niall and Nick Explanation
Frustrated in Some Parts	Unravelling a Tangled Series of Things	Saw Difficult Questions Solved Easily Motivated Me

Shared Theme
   Category
      Code
         Subcode
	Shared Theme
	   Category
	      Code
	         Subcode
		Shared Theme
		   Category
		      Code
		         Subcode

Frustration Becomes Enjoyable	Watching Compound Lines of Working Simplify	